

# A Model of Secular Stagnation: Theory and Quantitative Evaluation\*

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## Abstract

This paper formalizes and quantifies the secular stagnation hypothesis, defined as a persistently low or negative natural rate of interest leading to a chronically binding zero lower bound. Output-inflation dynamics and policy prescriptions are fundamentally different than in the standard New Keynesian framework. Using a 56-period quantitative lifecycle model, a standard calibration to US data delivers a natural rate ranging from  $-1.5\%$  to  $-2\%$ , implying an elevated risk of ZLB episodes for the foreseeable future. We decompose the contribution of demographic and technological factors to the decline in interest rates since 1970 and quantify changes required to restore higher rates.

Keywords: Secular stagnation, monetary policy, zero lower bound

JEL Classification: E31, E32, E52

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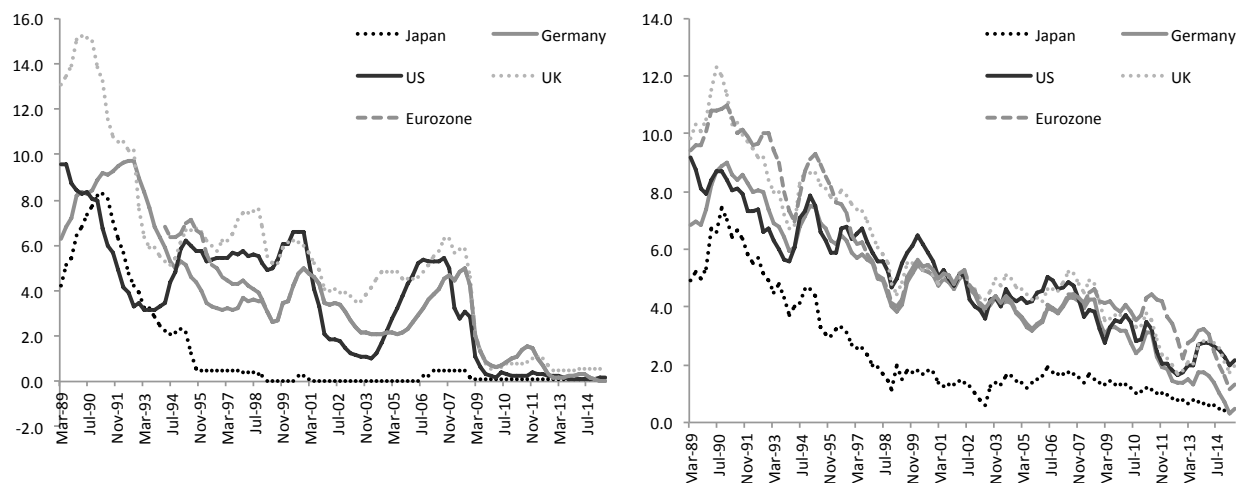
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**Figure 1: Short and Long-Term Interest Rates**



## 1 Introduction

The zero lower bound (ZLB) on the short-term nominal interest rate became a binding constraint in the United States in 2008 in the midst of the financial crisis. Accordingly, low interest rates are often tied to that event, and it may seem natural to presume that as the crisis of 2008 moves into the rear view mirror so will the low interest rate environment. This perspective seems vindicated by the recent interest rate increases by the Federal Reserve in Q4 2016. However, as shown in Figure 1, the low interest rate in 2008 was not just an anomaly that only arose due to the financial crisis. Instead, it is the culmination of a 25 year trend across major industrial economies. In Japan, rates have been zero since the mid 1990s and remain there still today. Furthermore, while recent increases in the Federal Funds rate may give rise to optimism in the US, rates remain at zero in Europe with no end in sight. The fact that rates remain so low should raise concern that the ZLB will remain a significant constraint on monetary policy in the US in the future should there be any deterioration in economic conditions.

This paper contemplates the possibility that the fall in interest rates observed over the past 25 years in the industrialized world represents a permanent change - a "new normal." We propose a model that allows for ZLB episodes of arbitrary duration due to a persistent fall in interest rates (as observed in Figure 1) driven by slow moving secular forces that are unlikely to reverse themselves. We also perform a quantitative assessment to ascertain the plausibility of a decline in the natural (full-employment) interest rate to negative values using a quantitative lifecycle model calibrated to current US data.

This paper should be read as a formalization of the secular stagnation hypothesis, originally proposed by Hansen (1939), who, a decade into the Great Depression with interest rates still at

zero, argued that the US economy faced a permanent shortfall in aggregate demand. This idea was recently resurrected by Lawrence Summers who argued that one should think of secular stagnation as the hypothesis that the natural rate of interest (the equilibrium real interest rate consistent with output at potential) is permanently negative (see [Summers \(2013\)](#) and [Summers \(2014\)](#)). We formalize this idea by constructing a series of analytic and quantitative overlapping generation models (OLG) of varying degrees of complexity in which the steady state, full employment real interest rate is permanently negative. This leads under certain conditions to a chronically binding ZLB, subpar growth, and inflation below target; we define these characteristics as a secular stagnation.

The policy implications of our framework differ in important ways from the standard ZLB literature. Perhaps most importantly, a policy of simply waiting for a ZLB episode to end is not a good strategy in a secular stagnation; there is no *deus ex machina* for recovery as presumed in the existing literature (see [Krugman \(1998\)](#) and [Eggertsson and Woodford \(2003\)](#) where the exogenous shocks that give rise to the ZLB must ultimately revert). The absence of an automatic recovery has fundamental implications for monetary policy. Unlike [Eggertsson and Woodford \(2003\)](#), forward guidance loses most of its power, for two reasons. First, agents do not necessarily anticipate a future date at which the ZLB is not binding. Second, aggregate demand (or the intertemporal IS equation) is not as “forward-looking” - sensitive to expectations about future income - in our setting relative to the standard model. Raising the inflation target may be an option to accommodate a negative natural rate of interest. This policy, however, suffers from two major drawbacks. First, an increase in the inflation target must be sufficiently large. For example, if the natural rate of interest is  $-4\%$ , the inflation target must be  $4\%$  or higher. Small changes in the inflation target have no effect, capturing Krugman’s observation of the “law of the excluded middle” or “timidity trap” when trying explain why the Japanese economy might not respond to a higher inflation target announced by the the Bank of Japan unless it was sufficiently aggressive (see [Krugman \(2014\)](#)). Second, even with a high enough increase in the inflation target, the secular stagnation equilibrium is not eliminated. Hence, an increase in the inflation target only *allows* for a better outcome, but it does not guarantee it as it cannot *exclude* a secular stagnation equilibrium.<sup>1</sup>

Fiscal policy is more effective in addressing the problems raised by secular stagnation and aggressive enough fiscal expansion does not suffer from multiplicity of steady states - it eliminates the secular stagnation steady state altogether. However, the effects of fiscal policy are more subtle than in the standard New Keynesian treatment of the ZLB. Increases in government spending can carry zero or negative multipliers in our model depending on the distribution of taxes across generations. The key for successful fiscal policy is that it must reduce the oversupply of savings and raise the natural rate of interest. Fiscal policy that instead increases desired savings by, for exam-

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<sup>1</sup>In fact, theory provides no obvious way to determine whether the high or low steady state will be chosen as they both pass standard equilibrium selection tests, such as local determinacy.

ple, reducing future disposable income through tax increases, can exacerbate a secular stagnation. In the standard NK model, government spending multipliers are above one irrespective of the financing mechanism and may be unboundedly large for reasons related to the forward-looking intertemporal IS curve. These mechanisms are considerably attenuated in our framework.

In addition to allowing for the possibility of arbitrarily long periods of negative interest rates, our model accounts for and can quantify a host of new forces that affect the natural rate of interest. These forces come naturally into play in our analysis since we abandon the representative agent framework of the standard NK model. Essentially any force that alters the relative supply of savings and investment can have an effect on the interest rate. We show how a slowdown in population growth or an increase in life expectancy puts downward pressure on the natural rate. Rising income inequality or a fall in the relative price of investment goods may also reduce the natural rate. A slowdown in productivity growth can also play an important role.<sup>2</sup>

Many existing narratives of the Great Recession in the US have emphasized the effect of debt deleveraging caused by the housing crisis on restraining aggregate demand and causing the ZLB to bind (see, for example, [Mian and Sufi \(2014\)](#), [Eggertsson and Krugman \(2012\)](#), [Guerrieri and Lorenzoni \(2011\)](#), and [Mehrotra \(2017\)](#)). For this reason, we incorporate financial frictions in our baseline model. In earlier models, even a permanent shock that tightens credit to financially constrained households results in only a temporary ZLB episode. Once the deleveraging cycle runs its course, the natural rate of interest rises, aggregate demand rises, and the ZLB episode ends. In stark contrast, in our model, a deleveraging cycle need not culminate in a rise in the natural rate, and there is no guarantee of a recovery once deleveraging ends. Indeed, our model predicts that, as the deleveraging cycle runs its course, interest rates continue to fall as the young borrowers who transition to saving in middle-age have greater disposable income. Incorporating financial frictions in our model also puts some structure on the idea that secular stagnation conditions may have existed prior to 2008 but were masked by the tech bubble in the late 1990s and the subsequent housing bubble in the early 2000s as suggested by [Summers \(2014\)](#).

While the first main contribution of this paper is an analytic framework that lays out the theoretical ingredients needed to characterize secular stagnation, the second main contribution is building a medium scale lifecycle model to explore whether persistently negative natural interest rates are quantitatively realistic. We build a 56 period OLG model with capital, and calibrate it to match the US economy in 2015, assuming that the output gap at that time is either (i) zero, or (ii) corresponds to the deviation of output from its pre-2008 trend - two natural (and extreme) benchmarks. In our calibrated model, this gives us a range for the natural rate of interest from  $-1.47\%$  to  $-2.20\%$ .

Our quantitative model is able to generate a permanently *negative* natural rate of interest using

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<sup>2</sup>It should be noted that the elasticity of the interest rate to changes in productivity growth differs in an OLG setting relative to the standard representative agent model.

parameters that are standard from the macro literature and matches key moments from the US data. The main drivers of negative natural interest rates are an aging population, low fertility, and sluggish productivity growth. While this trend may reverse itself, if current projections for fertility and productivity hold, our analysis suggests that the natural rate of interest will be low or negative for the foreseeable future.<sup>3</sup> Though productivity growth has experienced unexpected periods of acceleration and deceleration since the 1970s, the demographic factors accounting for a low natural rate of interest are unlikely to abate.

We use our model to understand the decline in interest rates seen in the data. We take our 2015 calibration and revert observable demographic and productivity factors to their 1970s value. Over this period, our model generates a 4.02% decrease in the real interest rate from 1970 to 2015 which matches the actual decrease of 4.09% experienced in the real Federal Funds rate over that time period. The reduction in fertility, mortality, and the rate of productivity growth play the largest role; each alone can account for a fall in the real interest rate of  $-1.84\%$ ,  $-1.92\%$  and  $-1.90\%$  percent respectively. The main factor which has tended to counterbalance these forces is an increase in government debt, which accounts for a 2.11% increase in the real interest rate. Changes in the labor share, the relative price of investment goods, and variation in consumer debt capacity play a quantitatively smaller role (the first two decrease rates by  $-0.50\%$  and  $-0.44\%$  while the latter raises rates by  $0.13\%$ ).

We also evaluate quantitatively under what assumptions one should expect real interest rates to revert to a more normal level of 1 percent - the assumption maintained by current Federal Reserve projections. At a 1% steady state real interest rate, the ZLB is much less likely to pose a problem for business cycle stabilization (see, for example, Williams (2016)). A key determinant of whether interest rates are likely to increase is whether the rate of productivity growth, which has slowed markedly since the 1970s, returns to its long run rate of 2% per year. This experiment makes clear that the lively debate between Robert Gordon and others about the likely future evolution of productivity is crucial in determining whether secular stagnation will remain a problem. Gordon (2012) takes a very pessimistic view about the evolution of future productivity, while Brynjolfsson and McAfee (2014), for example, take a more optimistic view. Our simulations suggest the stakes are high in that debate for the future conduct of monetary policy, as it may determine the extent to which the ZLB remains an issue for macroeconomic stabilization.

Our paper is organized as follows. In Section 2, we relate the paper to the existing literature. In Section 3, we present our baseline model starting with a simple endowment economy to explore interest rate determination in an OLG model with no nominal frictions. Section 4 adds price

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<sup>3</sup>Of course, a negative natural rate of interest in steady state does not exclude the possibility that we see a short-term rise in the nominal interest rate due to temporary, business cycle, factors. Instead it suggests that there are plausible conditions under which one should expect recurrent and chronic ZLB episodes going forward that can be of arbitrary duration.

level determination. Sections 5 and 6 incorporates nominal rigidities and shows how negative real interest rates can create a demand recession. Section 7 evaluates monetary and fiscal policy responses in a secular stagnation. Section 8 presents a 56 generation lifecycle model with capital accumulation and borrowing constraints to allow for a quantitative evaluation of the secular stagnation hypothesis.

## 2 Related Literature

It may seem somewhat surprising that the idea of secular stagnation has not already been studied in detail in the relatively large literature on the liquidity trap. This literature already invites the possibility that the zero bound on the nominal interest rate is binding for some period of time due to a drop in the natural rate of interest. The reason for this omission, we suspect, is that secular stagnation does not emerge naturally from the current vintage of models in use in the literature. This, however - and perhaps unfortunately - has less to do with economic reality than with the limitations of these models. Most analyses of the current crisis take place within representative agent models (see e.g. Krugman (1998), Eggertsson and Woodford (2003), Christiano, Eichenbaum and Rebelo (2011) and Werning (2012) for a few well known examples) where the long run real interest rate is directly determined by the inverse of the discount factor of the representative agent. Zero lower bound episodes are caused by temporary shocks to the discount factor triggering temporary reductions in the real interest rate that eventually must revert back to a positive long-run level.<sup>4</sup> The second generation of these models as in Eggertsson and Krugman (2012) that more explicitly incorporate financial frictions and deleveraging cannot generate permanently negative rates either since the steady state real interest rate is tied to the discount rate of the representative saver.

Our model differs fundamentally from this earlier generation of models. As has been understood since Samuelson (1958), with an OLG structure, the discount factor of a representative saver is no longer the sole determinant of the natural rate of interest, and a negative natural rate that lasts for an arbitrarily long time is now a possibility.<sup>5</sup> Our contribution is to introduce the zero lower bound in a framework where the natural rate of interest is influenced by a broader set of factors than in the typical infinite-horizon model. As we argue, these differences are nontrivial for thinking about monetary and fiscal policy at the zero lower bound. Furthermore, an OLG structure allows for a richer set of factors to influence the real interest rate which is essential for

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<sup>4</sup>A permanent shock to the household's discount factor is not possible since the maximization problem of the representative household is no longer well defined. One possible alternative in the representative agent framework is a large enough rise in uncertainty to make the risk-free rate negative as in Abel et al. (1989).

<sup>5</sup>For a more recent reference that provides a good overview of the results in the literature, see Blanchard and Weil (2001).

understanding the sources and potential persistence of the decline in the natural rate.

A recent literature has emphasized that existing New Keynesian models of the zero lower bound tend to generate inflation and output dynamics at odds with recent ZLB episodes; NK models predict a sharp collapse in output and deflationary spirals for particularly long lasting ZLB episodes (see e.g. [Cochrane \(2016\)](#)). By contrast, our model does not suffer from these counterfactual dynamics. In a secular stagnation steady state, inflation is persistently below target and output falls below trend, which matches the observed dynamics of output, inflation and interest rates in the US, Eurozone, and Japan. Similarly, our model also does not suffer from the so called “forward guidance puzzle” - the idea that interest rate changes very far in the future have implausibly large effects on output and inflation today as discussed in [Del Negro, Giannoni and Patterson \(2012\)](#). The intertemporal IS equation in our model is not as forward-looking exhibiting endogenous discounting as in [McKay, Nakamura and Steinsson \(2016\)](#) due to lifecycle considerations as opposed to occasionally binding borrowing constraints in their model.

A literature related to our work that explores the deflation steady state in a standard NK model is discussed in [Benhabib, Schmitt-Grohé and Uribe \(2001\)](#), [Schmitt-Grohé and Uribe \(2017\)](#) and [Benigno and Fornaro \(2015\)](#). A key difference to this strand of the literature is that, in these papers, the ZLB is binding due to self-fulfilling expectations rather than the fundamental factors we consider here. Moreover, the ZLB steady state in these models is locally indeterminate. In contrast, the secular stagnation equilibrium in our model satisfies normal determinacy conditions. This allows us to consider well defined comparative statics, and our steady state is immune from criticism as in [Christiano, Eichenbaum and Johannsen \(2016\)](#) who argue that indeterminate ZLB episodes driven by self-fulfilling expectations can be ruled out on the grounds that they are not “learnable” and hence it is unclear how expectations can coordinate on this steady state (see [Gibbs \(2017\)](#)).

Other closely related work includes [Kocherlakota \(2013\)](#) who also considers a permanent liquidity trap. The focus in that paper is on falling land prices as a trigger for the crisis, while our focus is on the forces usually associated with secular stagnation: population dynamics, income inequality, a decline in the relative price of investment goods, and a debt deleveraging shock.<sup>6</sup> Likewise, [Caballero and Farhi \(2014\)](#) characterize a long lasting ZLB episode due to a scarcity of safe assets. This work is more closely focused on explaining falling rates during the Great Recession as opposed to long-term interest rate trends that is the focus of the secular stagnation hypothesis. It is worth stressing, however, that the safety trap mechanism is not inconsistent with the forces we emphasize as driving secular stagnation. Indeed, building on an earlier version of this paper, [Ca-](#)

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<sup>6</sup>In the context of the Japanese crisis, [Krugman \(1998\)](#) suggested that population dynamics might be driving some of the decline in the natural rate of interest, although he did not explore this possibility explicitly. [Carvalho and Ferrero \(2014\)](#) quantify this force in a medium-scale DSGE model in the Japanese context and argue that demographic pressures were a significant contributor. Another recent contribution in the same vein focusing on US age dynamics is [Gagnon, Johannsen and Lopez-Salido \(2016\)](#).



ballero and Farhi (2014) explicitly incorporate the supply side of our model and show that many of the same policy conclusions apply. In separate work, the simple model in this paper is extended to an open economy setting (Eggertsson et al. (2016)) where the open economy dimensions of secular stagnation are analyzed. Open economy factors are particularly relevant when thinking about the global savings glut identified in Bernanke (2005) as a source of low interest rates in the early 1990s as well as understanding policy spillovers from current account imbalances and monetary/fiscal policy interactions across countries in a global secular stagnation.

### 3 Endowment Economy

We start by considering a simple overlapping generations economy to analyze the determination of interest rates in an OLG setting. Households live for three periods. They are born in period 1 (young), enter middle age in period 2 (middle age), and retire in period 3 (old). Consider the case in which no aggregate saving is feasible (that is, there is no physical capital), but that generations can borrow and lend to one another. Moreover, imagine that only the middle and old generation receive any income in the form of an endowment:  $Y_t^m$  and  $Y_t^o$ . In this case, the young will borrow from the middle-aged households, which, in turn, will save for retirement. The old will not save, but will fully consume their remaining income and assets. We assume, however, that there is a limit on the amount of debt the young can borrow. Generally, we would like to think of this as reflecting some sort of incentive constraint, but for the purposes of this paper, it will simply take the form of an exogenous time-varying constant  $D_t$  (as in the “debtors” in Eggertsson and Krugman (2012)).

More concretely, consider the representative household of a cohort born at time  $t$ . This household has the following utility function:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o} \mathbb{E}_t \{ \log(C_t^y) + \beta \log(C_{t+1}^m) + \beta^2 \log(C_{t+2}^o) \}$$

where  $C_t^y$  is the consumption of the household when young,  $C_{t+1}^m$  its consumption when middle aged, and  $C_{t+2}^o$  its consumption while old. We assume that borrowing and lending take place via a one period riskless bonds denoted  $B_t^i$  where  $i = y, m, o$  at an interest rate  $r_t$ . Given this structure, we can write the budget constraints facing households born at time  $t$  in each period as:

$$C_t^y = B_t^y \tag{1}$$

$$C_{t+1}^m = Y_{t+1}^m - (1 + r_t) B_t^y + B_{t+1}^m \tag{2}$$

$$C_{t+2}^o = Y_{t+2}^o - (1 + r_{t+1}) B_{t+1}^m \tag{3}$$

$$(1 + r_t) B_t^i \leq D_t \tag{4}$$

where equation (1) corresponds to the budget constraint for the young where consumption is financed by borrowing. Equation (2) gives the budget constraint of the middle-aged household



who receives the endowment  $Y_t^m$ , repays what was borrowed, and saves  $B_{t+1}^m$  for retirement. Finally, equation (3) corresponds to the budget constraint when the household is old, consuming savings and interest and any endowment received in the last period.<sup>7</sup>

The inequality (4) corresponds to the exogenous borrowing limit (as in Eggertsson and Krugman (2012)). We assume that the collateral constraint is tight enough so that it binds for the young:<sup>8</sup>

$$C_t^y = B_t^y = \frac{D_t}{1 + r_t} \quad (5)$$

where the amount of debt that the young can borrow depends on their ability to repay in the middle period and, therefore, includes interest payments. As a result, a drop in the real interest rate increases borrowing by the young.

The old at any time  $t$  are hand to mouth and will consume their income:

$$C_t^o = Y_t^o - (1 + r_{t-1}) B_{t-1}^m \quad (6)$$

The middle aged, however, are at an interior solution and their consumption-saving choices satisfies a standard Euler equation given by:

$$\frac{1}{C_t^m} = \beta \mathbb{E}_t \frac{1 + r_t}{C_{t+1}^o} \quad (7)$$

We assume that the size of each generation is given by  $N_t$ . Let us define the growth rate of the new cohort by  $1 + g_t = \frac{N_t}{N_{t-1}}$ . Equilibrium in the bond market requires that borrowing of the young equals the savings of the middle aged so that  $N_t B_t^y = -N_{t-1} B_{t-1}^m$  or:

$$(1 + g_t) B_t^y = -B_{t-1}^m \quad (8)$$

An equilibrium is now defined as a set of stochastic processes  $\{C_t^y, C_t^o, C_t^m, r_t, B_t^y, B_t^m\}$  that solve (1), (2), (5), (6), (7), and (8) given an exogenous process for  $\{D_t, g_t\}$ .

To analyze equilibrium determination, let us focus on equilibrium in the market for savings and loans given by equation (8) using the notation  $L_t^s$  and  $L_t^d$ ; the left-hand side of (8) denotes the demand for loans,  $L_t^d$ , and the right-hand side its supply,  $L_t^s$ . Hence the demand for loans (using (5)) can be written as:

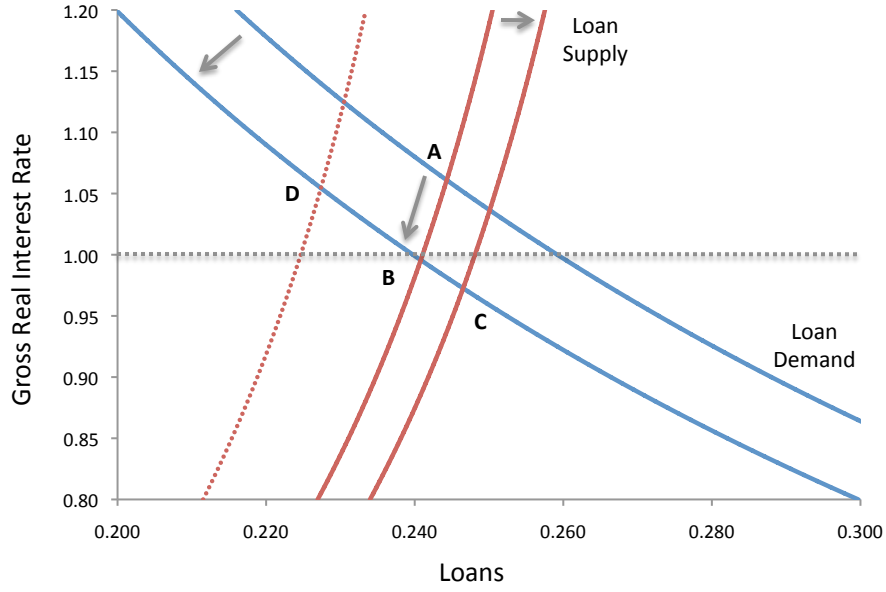
$$L_t^d = \frac{1 + g_t}{1 + r_t} D_t \quad (9)$$

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<sup>7</sup>The addition of a warm glow bequest motive would not affect our qualitative results as the steady state real interest may still remain negative. Thwaites (2015) shows that bequests do not materially impact determination of real interest rates in an OLG setting. However, in the full lifecycle model, we include an explicit bequest motive.

<sup>8</sup>For the constraint to be binding, it must be the case that  $\frac{Y_t^m + Y_{t+1}^o / (1 + r_t)}{1 + \beta(1 + \beta)} > D_{t-1}$ . We check in our numerical experiments that the relevant condition is satisfied.

Figure 2: Equilibrium in the asset market



while an expression for loan supply, assuming perfect foresight, can be derived by combining the household budget constraints and the middle generation Euler equation:

$$L_t^s = \frac{\beta}{1 + \beta} (Y_t^m - D_{t-1}) - \frac{1}{1 + \beta} \frac{Y_{t+1}^o}{1 + r_t} \quad (10)$$

The real interest rate, depicted in Figure 2, is then determined by the intersection of the loan demand,  $L_t^d$  and loan supply,  $L_t^s$ :

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t) D_t}{Y_t^m - D_{t-1}} + \frac{1}{\beta} \frac{Y_{t+1}^o}{Y_t^m - D_{t-1}} \quad (11)$$

In contrast to the standard representative agent model, the real interest rate will now, in general, depend on a host of factors in addition to the discount factor; the income profile over the lifecycle, the debt limit, and population growth all influence the real interest rate.

### 3.1 Productivity, Population Growth and Inequality

In our OLG setting, any factor that affects the relative supply or demand for loans changes the interest rate. Unlike in the standard representative agent model used in business cycle analysis, these forces can have permanent effects on the interest rate, and we should expect these dynamics to play out over an extended period of time. Importantly, there is nothing that prevents the real interest rate in expression (2) to be either positive or negative. Below we offer a few examples of the forces that may affect the equilibrium interest rate; this list is far from exhaustive, and in the quantitative section we take a firmer stand on which of these forces best account for declining US interest rates.

Let us first consider a potential fall in total factor productivity growth, which has been commonly associated with discussions of secular stagnation (this hypothesis is most forcefully articulated by [Gordon \(2015\)](#)). For this exercise, we assume that the income of the middle aged and the old is proportional to the aggregate endowment  $Y_t$ , which in turn is proportional to productivity so that  $Y_t = A_t \tilde{Y}$ . Moreover, as the debt limit reflects the extent to which the middle age agents can replay their debt, we assume that it grows with the middle aged income so that the debt limit relevant to the young at time  $t$  is given by  $D_t = A_{t+1} \tilde{D}$ . The loan demand and supply can then be written in terms of renormalized variables as shown.<sup>9</sup>

Consider now the effect of a slowdown in productivity. First, lower productivity growth shifts out the supply of savings since lower expected future income induces the middle generation to increase retirement savings. Second, the expectation of lower future productivity tightens the borrowing constraint of the young, leading to a backward shift in the demand for loans. The new equilibrium is shown in point C in Figure 2, which now depicts the renormalized variables. It should be noted that unlike in the representative agent model, it is not required for productivity growth to be negative in order for the real interest rate to become negative.<sup>10</sup> Instead, interest rates depend on how income is distributed over the lifecycle, as well as the interaction between productivity growth and the income distribution.

The mechanism by which a reduction in population growth lowers the interest rate is straightforward and can be seen directly by inspecting the expression for loan demand. As the number of young decreases relative to the middle age (a decline in  $g_t$ ), loan demand falls shifting back the  $L_t^d$  curve and lowering the real interest rate to point B in Figure 2. In the quantitative model we will also consider changes in mortality risk which has a relatively intuitive effect on the real interest rate. As mortality risk decreases, individuals save more for retirement, increasing the supply of saving and lowering the interest rate.

Our model can also be used to consider the impact of an increase in inequality on interest rates. Generically, the effect of an increase in inequality is ambiguous. However, there are plausible conditions, however, under which higher inequality will in fact reduce the natural rate of interest.

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<sup>9</sup>The demand and supply for loans can now be written as:

$$\tilde{L}_t^d = \frac{1 + g_t}{1 + r_t} \frac{A_t}{A_{t-1}} \tilde{D} \quad (12)$$

$$\tilde{L}_t^s = \frac{\beta}{1 + \beta} \left( \tilde{Y}^m - \tilde{D} \right) - \frac{1}{1 + \beta} \frac{\tilde{Y}^o}{1 + r_t} \frac{A_{t+1}}{A_t} \quad (13)$$

where *tilde* denotes that the original variable has been divided by productivity.

<sup>10</sup>In the representative household model (without population growth) the real interest rate is given by:

$$1 + r_t = \beta^{-1} \frac{A_{t+1}}{A_t}$$

so productivity would need to decline at a rate greater than the inverse of the discount factor for the interest rate to be negative. While productivity growth is low in many advanced countries, it is not negative.

We provide one such illustrative example in Appendix B, in which a fraction of the middle aged population are credit constrained. In this case shifting income from the low skilled, credit constrained households to the high skilled in middle age reduces the real interest rate. In general, the condition needed for inequality to reduce rates is that those with higher incomes at a given age save more than those with lower incomes.<sup>11</sup>

A fourth force often associated with secular stagnation is a persistent fall in the relative price of investment goods. As investment goods become cheaper, less savings is needed to finance a given level of investment in the capital stock thereby reducing overall demand. Since our illustrative example does not include capital, we will defer this discussion to Section 8.

### 3.2 Deleveraging

One of the main narratives about the source of the crisis of 2008 is that it was triggered by a debt deleveraging shock (see e.g. Mian and Sufi (2014)). Household took on excessive levels of mortgage debt in the mid 2000s and the 2008 crisis corresponded to an abrupt correction where household were forced to pay down high levels of debt - often referred to as a "Minsky moment". Eggertsson and Krugman (2012) (hereafter EK) model this as permanent tightening in the household collateral constraint  $D_t$ . A key prediction of EK and related models is that once agents in the economy pay down their debt to the new sustainable level - the debt deleveraging ends - interest rates return to their precrisis level. However, in our framework, this prediction is overturned, and the end of a debt deleveraging cycle does not lead to a normalization of interest rates.

Point B in Figure 2 shows the equilibrium level of the real interest rate on impact after a deleveraging shock. As we can see, the shock leads directly to a reduction in the demand for loans since the demand curve shifts inward. The supply of loanable funds is unchanged since the debt repayment of the middle generation depends on the lagged value of collateral constraint,  $D_{t-1}$ .

On impact, the young are now spending less at a given interest rate, while the middle aged and old are spending the same. Since the endowment must be fully consumed in our economy, this fall in spending by the young then needs to be made up by inducing some agents to spend more. This adjustment takes place via a reduction in the real interest rate that induces more spending.<sup>12</sup>

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<sup>11</sup>Most realistic models of bequests, for example, argue that the preference for leaving a bequest increases with income, implying an increase in bequests with higher inequality. See Dynan, Skinner and Zeldes (2004) for a discussion of other mechanisms, such as the persistence of skill types across generations. We have also experimented with a production structure where skill-bias technological change (as in Krusell et al. (2000)) increases inequality and found plausible conditions under which skill-biased technical change puts downward pressure on the real interest rate. We leave a fuller analysis for future research.

<sup>12</sup>The drop in the real interest rate stimulates spending via two channels: First, as equation (11) shows, a fall in the real interest rate makes consumption today more attractive to the middle aged, thus increasing their spending. Observe that the  $L_t^s$  curve is upward sloping in the real interest rate; in general, the slope of the savings curve depends upon the

So far, the mechanism described in our model is exactly the same as in EK. A deleveraging shock triggers a drop in spending by borrowers at the existing rate of interest. The real interest rate then falls to keep the aggregate level of spending the same.

In EK, the economy reaches a new steady state in the next period in which, once again, the real interest rate is determined by the discount factor of the representative savers in the economy. In that setting, the loan supply curve shifts back so that the real interest rate is exactly the same as before (as seen in point D). Loan supply shifts back in EK because borrower deleveraging reduces interest income accruing to savers which implies that their supply of savings falls in equilibrium. In contrast, in our model there is no representative saver; instead, households are both borrowers and savers at different stages in their lives. The fall in the borrowing of young households in period  $t$  then implies that in the next period - when that agent becomes a saver - each household has more resources to save since disposable income is higher. At time  $t + 1$ , the supply of savings  $L_t^s$  shifts outwards as shown in Figure 2. In sharp contrast to EK, where the economy settles back on the old steady state after a one-period transition, the economy in our model reaches a new steady state with a permanently lower real rate of interest. The new interest rate may be negative, depending on the size of the shock. This process can serve as a powerful and persistent propagation mechanism for the original deleveraging shock. More generally, even if the drop in  $D_t$  is not permanent, the natural rate of interest will inherit the dynamics of the drop in  $D_t$ , which may be of arbitrary duration. One interesting implication of these dynamics is that policies such as macroprudential policies that are implemented in the midst of a deleveraging crisis run the risk of permanently depressing the natural rate of interest.

## 4 Price Level Determination

We now introduce perfectly flexible nominal prices. We will show that this introduces a kink in the model: if the steady state equilibrium interest rate is negative, then there is no equilibrium consistent with inflation below a certain level. For example, if the real interest rate is  $-3\%$ , then inflation must be greater than  $3\%$  in a steady state that respects the zero lower bound on nominal interest rates. This will have fundamental implications when we introduce realistic nominal frictions; the unwillingness or inability of the central bank to accommodate a high enough inflation rate will result in a demand shortfall and output contraction.

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elasticity of intertemporal substitution. If preferences exhibit very weak substitution effects and if the endowment is received only by the middle-aged generation, it is possible to have a downward sloping  $L_t^s$  curve. However, our view is that the empirically relevant case is when savings are increasing in the interest rate. Second, for the credit-constrained young generation, a reduction in the real interest rate relaxes their borrowing constraint. A lower interest rate allows them to take on more debt,  $B_t^y$ , for any given value of  $D_t$ . Since borrowing is limited by their ability to repay in the next period, changes in the interest rate which affect the size of the total repayment affect borrowing today.

As is standard in the literature, we introduce a nominal price level by assuming that one period nominal debt denominated in money is traded, and that the government controls the nominal rate of return on this asset.<sup>13</sup> The saver in our economy (the middle-generation household) now has access to risk-free nominal debt which is indexed in dollars, in addition to one period risk-free real debt.<sup>14</sup> This assumption gives rise to a consumption Euler equation which is the nominal analog of the Euler equation (7):

$$\frac{1}{C_t^m} = \beta E_t \frac{1}{C_{t+1}^o} (1 + i_t) \frac{P_t}{P_{t+1}} \quad (14)$$

where  $i_t$  is the nominal rate and  $P_t$  is the price level. We impose a non-negativity constraint on nominal rates. Implicitly, we assume that the existence of money precludes the possibility of a negative nominal rate. At all times:

$$i_t \geq 0 \quad (15)$$

Equations (7) and (14) imply (assuming perfect foresight) the standard Fisher relation:

$$1 + r_t = (1 + i_t) \frac{P_t}{P_{t+1}} \quad (16)$$

where again the equilibrium real interest rate  $r_t$  is determined by equation (11). The Fisher equation simply states that the real interest rate should be equal to the nominal rate deflated by the growth rate of the price level (inflation).<sup>15</sup>

From (15) and (16), it follows that if the real rate of interest is permanently negative, there is no equilibrium consistent with a stable price level. To see this, assume there is such an equilibrium so that  $P_{t+1} = P_t = P^*$ . Then the Fisher equation implies that  $i_t = r_t < 0$ , violating the zero bound. Hence a constant price level - price stability - cannot be sustained when  $r_t$  is negative.<sup>16</sup>

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<sup>13</sup>There are various approaches to microfound a demand for money by using money in the utility function or cash-in-advance constraints. For completeness, we explicitly add money in the utility function to our model in Appendix F.

<sup>14</sup>For simplicity, we assume that this asset trades in zero net supply, so that, in equilibrium, the budget constraints already considered are unchanged. However, we relax this assumption once we incorporate fiscal policy. Strictly speaking what is needed, once money is explicitly incorporated as in Appendix F with explicit fiscal policy rules, is that net government nominal debt (inclusive of money) is in zero supply. As we show, our conclusions are unchanged when one considers a strictly positive level of government debt.

<sup>15</sup>Again, we can define an equilibrium as a collection of stochastic processes  $\{C_t^y, C_t^o, C_t^m, r_t, i_t, B_t^y, B_t^m, P_t\}$  that solve (1), (2), (5), (6), (7) and (8) and now in addition (14) and (15) given an exogenous process for  $\{D_t, g_t\}$  and a specification for monetary policy like an interest rate rule.

<sup>16</sup>For this result, it is important that we rule out bubbles and that fiscal policy keeps the real value of government liabilities constant. With passive fiscal policy and positive government liabilities, the price level could fall sharply on impact to raise the real value of debt to raise the natural rate of interest back to zero. After impact, the nominal rate will be at zero and the central bank will keep prices stable. This possibility will become more transparent once fiscal policy is explicitly introduced. We thank Jaume Ventura for pointing this out.

Let us denote the growth rate of the price level (inflation) by  $\Pi_t = \frac{P_{t+1}}{P_t} = \bar{\Pi}$ . The zero bound and the Fisher equation then implies that for an equilibrium with constant inflation to satisfy the ZLB, there is a bound on the inflation rate given by  $\Pi(1 + r) = 1 + i \geq 1$  or:

$$\bar{\Pi} \geq \frac{1}{1 + r} \quad (17)$$

which implies that steady state inflation is bounded from below by the real interest rate due to the zero bound. We term this *the natural lower bound on inflation*.

At a positive real interest rate the natural bound is of little relevance. If, as is common in the literature using representative agent models, the real interest rate in steady state is equal to the inverse of the discount factor, then the natural bound says that  $\Pi \geq \beta$ . Under a standard, calibration this implies a bound on the steady state inflation rate of about  $-2\%$  to  $-4\%$ . This is of course well below the inflation target of most central banks, making this bound of little empirical relevance.

With permanently negative real rates, however, this bound takes on a greater practical significance. If the real interest rate is negative, it implies that under flexible prices, steady state inflation must be positive. Suppose, for example, that the natural rate of interest is  $-4\%$  in Europe; this implies the natural lower bound on inflation is 4 percent. But what happens if the European Central Bank (ECB) refuses to allow inflation to rise above  $2\%$ ? With flexible prices, this possibility cannot be contemplated in the model; it does not admit an equilibrium. If the real interest rate is negative, the inflation rate *has to be positive* - the ECB must target inflation above  $4\%$  in the example above.

This is an unappealing result. A key question we want to ask our model is: what happens if the natural rate of interest is negative, and the central bank does not provide enough inflation, so that the zero lower bound becomes binding? So far, our model is not able to answer the question, as there is simply no equilibrium under this scenario. A straightforward solution, and one that fits the data, is to employ nominal rigidities in the price system. Once we introduce nominal frictions, then the central bank can indeed set inflation below the natural lower bound on inflation. In this case, however, the refusal of the central bank to allow inflation to rise to or overshoot its target results in an output gap with the nominal interest rate stuck permanently at zero.

## 5 Aggregate Supply

In this section, we incorporate nominal rigidities into our baseline model. The presence of nominal rigidities carry important implications for adjustment when the natural rate falls negative. We show that if the natural rate of interest is negative and the central bank does not set a sufficiently high inflation target, an employment and output gap emerges.

While it is common for models to feature short-term pricing rigidities, our model will incorporate a form of long run rigidity. Since our modeling choice is uncommon, some initial discussion



is warranted before we proceed to our exact specification. There is a broad consensus among economists that with permanently high levels of inflation, expectations about future inflation will ultimately adjust so that there is no long run tradeoff between inflation and unemployment. This empirical prediction was indeed largely borne out in the 1970s in the United States. Our model will incorporate this neutrality of high inflation in the form of a long run vertical aggregate supply curve when inflation is sufficiently high (see the top half of the aggregate supply curve in Figure 3).

In contrast, a similar consensus has not emerged of long run neutrality in a low inflation/deflation environment. Tobin (1972) argued that, during the Great Depression, firms were reluctant to cut nominal wages despite high unemployment. Tobin argued that this behavior suggests a permanent, long run tradeoff between inflation and unemployment at low levels of inflation. Others have built on and formalized Tobin’s analysis in the form of an upward sloping long run Phillips curve (see for example Akerlof, Dickens and Perry (1996), Kim and Ruge-Murcia (2009), Fagan and Messina (2009), Benigno and Ricci (2011), Coibion, Gorodnichenko and Wieland (2012) and Daly and Hobijn (2014)). We incorporate this notion of aggregate supply in our model leading to an upward sloping Phillips curve (see the bottom half of the aggregate supply curve in Figure 3).

Alternative microfoundations for introducing nominal rigidities will provide similar qualitative results as long as the models feature a long-run tradeoff between inflation and output.<sup>17</sup> Here, we opt for capturing this tradeoff by introducing downward nominal wage rigidity; this specification has the virtue of capturing the neutrality of inflation when it is high, yet simultaneously giving rise to meaningful tradeoffs at low inflation rates.

A large body of evidence documents the presence of downwardly nominally rigid wages, even in the face of high unemployment. This notion dates back as far as Malthus, who noted that “it very rarely happens that the nominal price of labour universally falls” (Malthus (1798)). Bewley (1999) interviewed firm executives directly and explicitly documented their reluctance to cut nominal wages. More recently, substantial nominal wage rigidity has been studied in US administrative data by Fallick, Lettau and Wascher (2011), in worker surveys by Barattieri, Basu and Gottschalk (2014), and in cross-country data by Schmitt-Grohé and Uribe (2016). Our specification is closely related to the supply side of Schmitt-Grohé and Uribe (2016), and our calibrated level of wage rigidity in our quantitative analysis (see Section 8) matches closely their estimates for the degree of wage rigidity.<sup>18</sup>

We simplify our exposition by assuming that only the middle generation receives income, now

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<sup>17</sup>In Appendix E we provide perhaps the most standard way of specifying rigidities, the New Keynesian Calvo model.

<sup>18</sup>Relative to their specification, our specification allows us to vary the degree to which wages are downwardly rigid or flexible and show that increasing flexibility does not lead to stabilization illustrating the basic failure of the price mechanism in restoring full employment in a secular stagnation. Also see Shimer (2012) and Kocherlakota (2013) for further discussion of how wage rigidities can explain labor dynamics the Great Recession.

as payment for labor supplied instead of as an exogenous endowment. The budget constraint for young agents is again given by equations (1) and (4), but now we replace the budget constraint of the middle generation (2) and old generation (3) with the following:

$$C_{t+1}^m = \frac{W_{t+1}}{P_{t+1}} L_{t+1} + \frac{Z_{t+1}}{P_{t+1}} - (1 + r_t) B_t^y + B_{t+1}^m \quad (18)$$

$$C_{t+2}^o = -(1 + r_{t+1}) B_{t+1}^m \quad (19)$$

where  $W_{t+1}$  is the nominal wage rate,  $P_{t+1}$  the aggregate price level,  $L_{t+1}$  the labor supply of the middle generation, and  $Z_{t+1}$  the profits of the firms. For simplicity, we assume that the middle generation will supply a constant level of labor  $\bar{L}$  inelastically. Note that if the firms do not hire all available labor supplied, then labor demand  $L_t$  may be lower than labor supply  $\bar{L}$  due to rationing. Under these assumptions, each of the generations' consumption-saving decisions remains the same as before.<sup>19</sup>

On the firm side, we assume that firms are perfectly competitive and take prices as given. They hire labor to maximize period-by-period profits. Their problem is given by:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t \quad (20)$$

$$\text{s.t.} \quad Y_t = L_t^\alpha \quad (21)$$

The firms' labor demand condition is then given by:

$$\frac{W_t}{P_t} = \alpha L_t^{\alpha-1} \quad (22)$$

So far we have described a perfectly frictionless production side, and, if this were the end, our model would be analogous to what we have already considered in the endowment economy. Output would be given by  $Y_t = L_t^\alpha = \bar{L}^\alpha$  and the labor demand condition (22) would fix the real wage.

Now, consider a world in which households will never accept working for wages that fall below their wage in the previous period, so that nominal wages at time  $t$  cannot be lower than what they were at time  $t - 1$ . Or, slightly more generally, imagine that the household would never accept lower wages than a wage norm given by  $\tilde{W}_t = \gamma W_{t-1} + (1 - \gamma) W_t^{flex}$  where:

$$W_t^{flex} = P_t \alpha \bar{L}^{\alpha-1} \quad (23)$$

If  $\gamma = 1$ , the wage norm is simply last period's nominal wages and wages are perfectly downwardly rigid; if  $\gamma = 0$ , we obtain the flex-price nominal wage. Formally, wages in our economy are given by

$$W_t = \max \left\{ \tilde{W}_t, W_t^{flex} \right\} \quad \text{where} \quad \tilde{W}_t = \gamma W_{t-1} + (1 - \gamma) W_t^{flex} \quad (24)$$

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<sup>19</sup>The rationing approach followed here and in Schmitt-Grohé and Uribe (2016) is essentially of the same form as in general disequilibrium model of Barro and Grossman (1971); for a review of this literature see Bénassy (1993).

Nominal wages in our economy can never fall below the wage norm  $\tilde{W}_t$ . If labor market clearing requires lower nominal wages than the previous nominal wage rate, then the labor market will not clear, and labor is rationed.<sup>20</sup>

To close the model, we specify a monetary policy rule. We posit that the central bank sets the nominal rate according to a standard Taylor rule:

$$1 + i_t = \max \left( 1, (1 + i^*) \left( \frac{\Pi_t}{\Pi^*} \right)^{\phi_\pi} \right) \quad (25)$$

where  $\phi_\pi > 1$ ;  $\Pi^*$  and  $i^*$  are parameters of the policy rule that we hold constant. This rule states that the central bank attempts to keep inflation at the inflation target  $\Pi^*$  and the nominal rate at  $i^*$  so long as the nominal rate implied by the rule is not constrained by the zero bound.<sup>21</sup>

It should be noted that the results we present in this paper do not depend on the particulars of the Taylor rule. We are interested in exploring what happens in an inflation targeting regime when the natural rate of interest is negative and the central bank's inflation target is low enough that it cannot be reached even at a zero nominal rate. A simpler way of thinking about the policy regime we have in mind - and equivalent one for our purposes to the Taylor rule specified above - is that we are assuming that the central bank will set inflation equal to its target  $\Pi^*$  at all times (without needing to concerns ourselves with how exactly this is accomplished) *except* for when this implies a negative nominal interest rate. In this case, the nominal interest rate is zero and inflation falls below target (see Eggertsson et al. (2016) for an example of this specification of the policy regime).<sup>22</sup>

Equilibrium in the economy with nominal rigidities is defined as follows:

**Definition 1.** A competitive equilibrium is a sequence of quantities  $\{C_t^y, C_t^o, C_t^m, B_t^y, B_t^m, L_t, Y_t, Z_t\}$  and prices  $\{P_t, W_t, W_t^{flex}, r_t, i_t\}$  that satisfy (1), (5), (6), (7), (8), (14), (15), (18), (20), (21), (22), (23), (24) and (25) given an exogenous process for  $\{D_t, g_t\}$  and initial values for  $W_{-1}$  and  $B_{-1}^m$ .

We start by characterizing the steady state of the model, which can be shown graphically based on two relationships between output and inflation in steady state: aggregate supply and aggregate demand. The aggregate supply specification of the model consists of two regimes: one in which

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<sup>20</sup>If labor market clearing requires higher nominal wages than the past nominal wage rate, this specification implies that nominal wages will rise equating labor demand supply with the real wage is given by (22) evaluated at  $L_t = \bar{L}$ .

<sup>21</sup>In the Appendix F, we introduce money explicitly into our model and show that our results are not affected so long as fiscal policy keeps consolidated government liabilities constant.

<sup>22</sup>The key assumption here is that, at the ZLB, we do not allow for the possibility that the inflation is higher than the inflation target. Essentially, the central would not tolerate such an outcome since it could always raise the nominal interest rate in that scenario. The only equilibria then consistent with the policy regime would be one in which inflation is below target because the central bank is constrained by the ZLB. Of course, there do exist equilibria consistent with higher inflation targets that do accommodate a negative natural rate of interest. We see these equilibria as those that emerge from different policy regimes which are willing to accept higher inflation target (see Section 7).

real wage equals the market clearing real wage (if  $\Pi \geq 1$ ), and the other when the bound on nominal wages is binding ( $\Pi < 1$ ). Intuitively, positive inflation in the steady state means that wages behave as if they are flexible, since nominal wages must rise to keep real wages constant. If  $\Pi \geq 1$ , then labor demand equals the exogenous level of labor supply  $\bar{L}$  defining the full employment level of output  $Y^f$ :

$$Y = \bar{L}^\alpha = Y^f \text{ for } \Pi \geq 1 \quad (26)$$

This is shown as a solid vertical segment in the AS curve in Figure 3.

If there is deflation in the steady state ( $\Pi < 1$ ), the wage norm binds ( $W = \tilde{W}$ ) and the real wage exceeds the market-clearing real wage.<sup>23</sup> We can then derive a relationship between output and inflation tracing the lower segment of the AS curve:

$$\frac{\gamma}{\Pi} = 1 - (1 - \gamma) \left( \frac{Y}{Y^f} \right)^{\frac{1-\alpha}{\alpha}} \text{ for } \Pi < 1 \quad (27)$$

Equation (27) is simply a nonlinear Phillips curve. The intuition is straightforward: as inflation increases, real wages fall and firms hire more labor. Importantly, this Phillips curve relationship is not a short-run relationship; instead, it describes the behavior of steady state inflation and output. The aggregate supply curve is shown in Figure 3, with the vertical segment given by (26) and the upward sloping segment given by (27) and the kink point at  $\Pi = 1$ .<sup>24</sup>

Turning to the aggregate demand relation, we again have two regimes: one in which the zero bound is not binding, the other in which it is binding. When the nominal rate is unconstrained, we obtain the AD curve by combining the real interest rate equation, Fisher relation, and monetary policy rule - equations (11), (16), (25) - to get:

$$Y = D + \frac{(1 + \beta)(1 + g)D\Gamma^*}{\beta} \frac{1}{\Pi^{\phi_\pi - 1}} \text{ for } i > 0 \quad (28)$$

where  $\Gamma^* \equiv (1 + i^*)^{-1} (\Pi^*)^{\phi_\pi}$  is the composite policy parameter in the monetary policy reaction function. The upper portion of the AD curve in Figure 3 depicts this relationship. As inflation increases, the central bank raises the nominal interest rate by more than one for one (since  $\phi_\pi > 1$ ), which, in turn, increases the real interest rate and reduces demand.

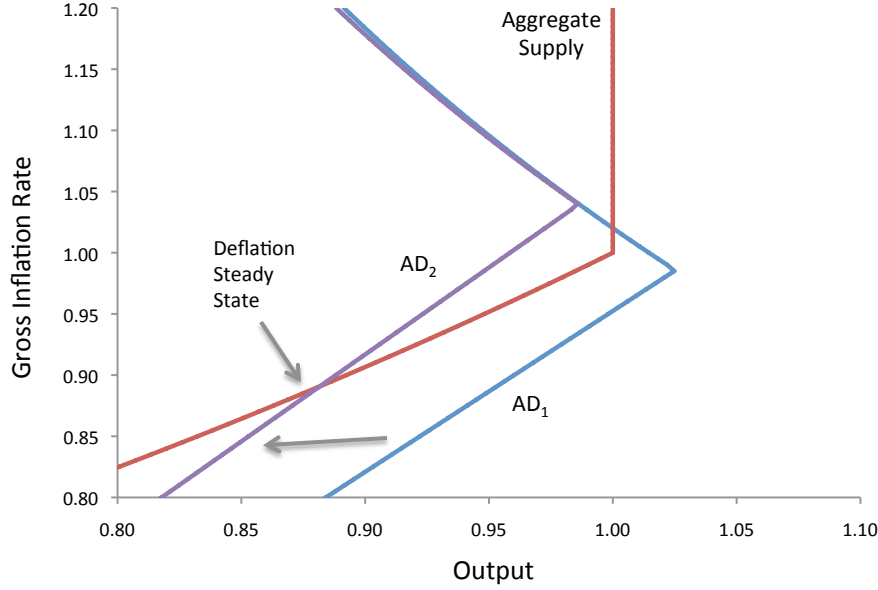
At the zero lower bound, we combine the same set of equations, but now impose  $i = 0$ . We obtain the following expression relating output and inflation:

$$Y = D + \frac{(1 + \beta)(1 + g)D}{\beta} \Pi \text{ for } i = 0 \quad (29)$$

<sup>23</sup>With a steady state deflation rate of  $\Pi$ , we can use equation 24 to find the state state wage rate of  $w = \frac{(1-\gamma)\alpha\bar{L}^{\alpha-1}}{1-\gamma\Pi^{-1}}$

<sup>24</sup>The AS and AD diagrams and numerical examples in this section assume the following parameter values:  $\beta = 0.985$ ,  $\gamma = 0.94$ ,  $\alpha = 0.7$ ,  $\Pi^* = 1.01$ ,  $\phi_\pi = 2$ ,  $D = 0.28$ ,  $g = 0.9\%$ .

**Figure 3:** Steady state aggregate demand and aggregate supply curves



In this case, the AD curve is upward sloping. The logic should again be relatively straightforward for those familiar with the literature on the liquidity trap: as inflation increases, the nominal interest rate remains constant, thus reducing the real interest rate. This change in the real rate raises consumption demand as shown by the bottom portion of the AD curve in Figure 3.

The kink in the aggregate demand curve occurs at the inflation rate at which monetary policy is constrained by the zero lower bound. That is, the AD curve depicted in Figure 3 will become upward sloping when the inflation rate is sufficiently low that the implied nominal rate the central bank would like to set is below zero. Mathematically, we can derive an expression for this kink point by solving for the inflation rate that equalizes the two arguments in the max operator of equation (25):

$$\Pi_{kink} = \left( \frac{1}{1 + i^*} \right)^{\frac{1}{\phi\pi}} \Pi^* \quad (30)$$

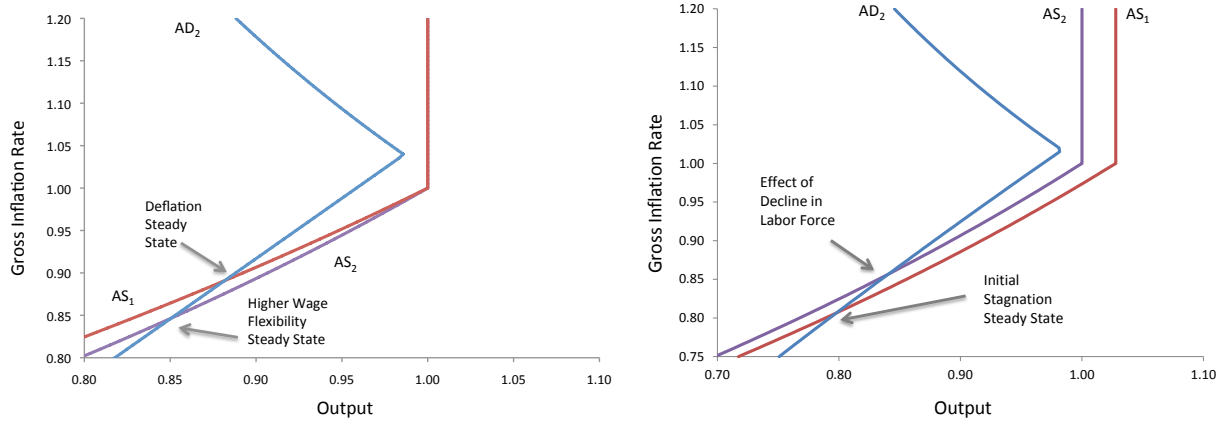
The location of the kink in the AD curve depends on both parameters of the policy rule: the inflation target  $\Pi^*$  and the targeted real interest rate  $i^*$ .

In what follows, it will be useful to define the natural rate of interest - the interest rate at which output is at its full employment level. The natural rate can be obtained by evaluating equation (11) at the full employment level of output  $Y^f$ :

$$1 + r_t^f = \frac{1 + \beta (1 + g_t) D_t}{\beta (Y^f - D_{t-1})}$$

It is useful to note that the full employment interest rate corresponds exactly to the real interest rate we derived in the endowment economy. Hence any of the forces that we showed affect the

**Figure 4:** Adjustment mechanisms: wage flexibility and hysteresis



real interest rate in the endowment economy will directly affect the full employment real interest rate in the more general setup.

## 6 Full Employment and Secular Stagnation

Equilibrium output and inflation is determined by the intersection of the aggregate demand and aggregate supply curves. We first consider a normal equilibrium when the natural rate of interest is positive. We assume that the central bank aims for a positive inflation target (that is,  $\Pi^* > 1$ ) and that the nominal interest rate target is consistent with the inflation target:  $1 + i^* = (1 + r^f)\Pi^*$ . With  $r^f > 0$ , the aggregate demand curve intersects the aggregate supply curve on the vertical segment of the AS curve. The exact intersection point is determined by the inflation target.<sup>25</sup> This full employment equilibrium is displayed in Figure 3. Under our assumed policy rule, the equilibrium depicted in Figure 3 is unique for a small enough inflation target and high enough  $\gamma$ .<sup>26</sup>

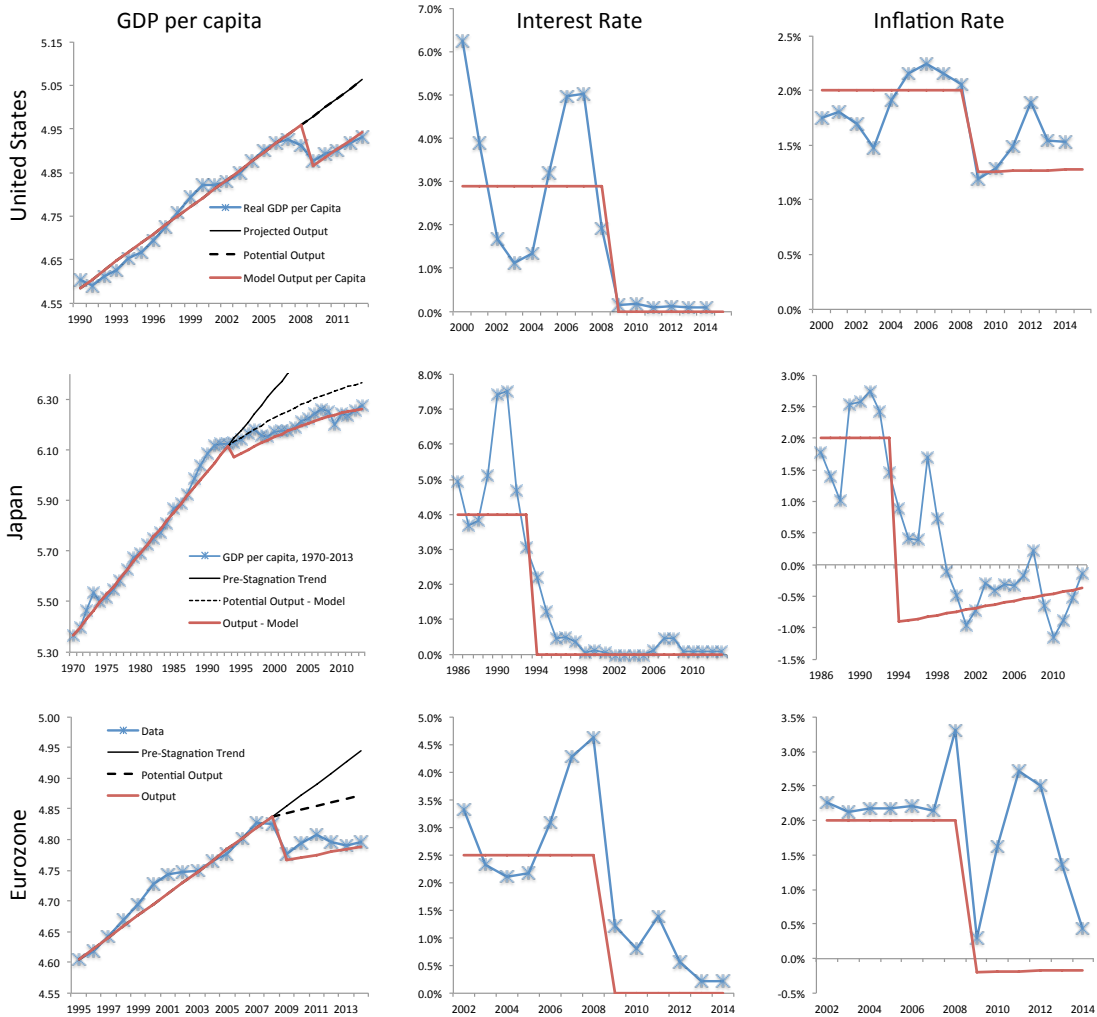
The making of a secular stagnation equilibrium is shown in Figure 3. Here, we illustrate the effect of a tightening in the collateral constraint  $D_t$ .<sup>27</sup> We assume the shock is large enough to move the natural rate of interest negative, and indeed below the inflation target:  $1 + r_t^f < (\Pi^*)^{-1}$ . As can be seen from equation (28), a reduction in  $D$  reduces output for any given inflation rate. This fall in output stems from the decline in consumption of the younger households, who cannot borrow as much as before to finance their consumption early in life. In the normal equilibrium, this drop in spending would be compensated by a drop in the real interest rate, restoring spending back to

<sup>25</sup>In case the central bank targets zero inflation -  $\Pi^* = 1$  - the intersection is at the kink of the AS curve.

<sup>26</sup>Uniqueness is guaranteed if  $\gamma = 1$ . As  $\gamma$  approaches zero, more equilibria are possible. We discuss these additional equilibria when they appear in a more general setting in Section 7.

<sup>27</sup>Or, equivalently, we could consider a fall in the rate of population or productivity growth.

**Figure 5:** Data v. model transition paths: US, Japan, Eurozone



its pre-shock level. However, the zero lower bound prevents this adjustment. Hence, the shock moves the economy off the full employment segment of the AS curve to a deflationary steady state where the nominal interest rate is zero. Here, steady state deflation raises steady state real wages above their market-clearing level thus depressing demand for labor and contracting output.<sup>28</sup>

**Proposition 1.** *If  $\gamma > 0$ ,  $\Pi^* = 1$ , and  $i^* = r^f < 0$ , then there exists a unique, locally determinate secular stagnation equilibrium.*

*Proof.* See Appendix D. □

As shown in Appendix C, if we linearize the model around the unique secular stagnation

<sup>28</sup>As we show, outright deflation is not central to the mechanism. If wages are indexed to the inflation target, then inflation that falls below target results in real wages that exceed the market-clearing level and output falls below the full-employment level.



steady state, the dynamic system is locally determinate. The determinacy of the secular stagnation steady state in our model stands in contrast to the indeterminacy of the deflation steady state analyzed in [Schmitt-Grohé and Uribe \(2013\)](#) and [Benigno and Fornaro \(2015\)](#).<sup>29</sup> Indeterminacy of equilibrium has been one justification for disregarding the deflation steady state and restricting analysis to the determinate positive interest rate steady state in New Keynesian models. As shown in Proposition 1, the AD curve must be steeper than the AS curve at the intersection point which implies the local determinacy condition. Importantly, local determinacy implies that an economy in secular stagnation may continue to experience well defined (unique bounded) business cycle fluctuations in unemployment and output around a permanently depressed growth path.

Our model delivers a permanent steady state slump with no pull towards full employment. A natural question is what forces can move the economy back to full employment, absent any government intervention. First, and perhaps obviously, if the shock that pushes the natural rate negative is only temporary, then the full employment equilibrium can be attained. To the extent that the shocks leading to negative real interest rates are slow moving (such as demographic factors) there is less reason to be optimistic about this adjustment mechanism.

A key friction in triggering unemployment was the fact that wages were downwardly rigid. A natural question, then, is whether an increase in the flexibility of the wage adjustment process will push the economy back to full employment. The answer to this question is, surprisingly, no. This result emerges because an increase in price/wage flexibility triggers a drop in expected inflation, increasing the real interest rate. This cannot be offset by interest rate cuts due to the zero bound. As wages become more flexible (a decrease in the parameter  $\gamma$ ), the slope of the AS curve steepens (see the left-hand panel of Figure 4). For any given deflation steady state, a decrease in  $\gamma$  will shift the steady state along the AD curve, increasing the rate of deflation, raising real wages, and therefore increasing the shortfall in output. This result echoes the paradox of flexibility shown in [Eggertsson and Krugman \(2012\)](#) and considered in a more general setting in [Bhattarai, Eggertsson and Schoenle \(2014\)](#).

Finally, a third adjustment mechanism is a reduction in the labor force by discouraged workers (or workers whose skills deteriorate) after prolonged spells of unemployment. Labor force participation fell markedly in the US and the duration of unemployment remains elevated despite recent reductions in the unemployment rate. A reduction in the labor force reduces downward pressures on wages and weakens deflationary pressure. If the contraction of the labor force proceeds far enough, the output gap is eliminated, with the AD curve intersecting the AS curve at the new, lower full employment level of output.<sup>30</sup>

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<sup>29</sup>For further discussion on the stability properties in [Benhabib, Schmitt-Grohé and Uribe \(2001\)](#), see [Bullard \(2010\)](#).

<sup>30</sup>Under hysteresis mechanisms, the AS curve shifts inwards along the transition path. Unemployment falls and inflation rises back toward target while nominal interest rates stay at the ZLB. Relative to the pre-stagnation trend, output remains depressed - in this extension, there is no catch up back to the pre-trend level of employment. Hysteresis

In Figure 5, we present data on the evolution of output, inflation and the short-term nominal interest rate in the US, Japan and the Eurozone. We calibrate our simple three period model and show how our model generates dynamics consistent with the observed behavior of these macro aggregates. The key message of the figure is that our model has, in principle, no difficulty generating a persistent fall in inflation, interest rates at the ZLB, and output persistently below trend. These features are in stark contrast to the standard New Keynesian model in which the model explodes (i.e. a determinate equilibrium no longer exists) if the natural rate of interest is expected to be negative for a long enough period (see e.g. Eggertsson and Singh (2016)). Even though highly stylized, a straightforward calibration of our simple model does a good job of capturing the transition dynamics of output and interest rates in these stagnation episodes. To conserve space, we defer the description of the calibration underlying Figure 5 to Appendix H; this allows us to focus on a richer quantitative lifecycle model.<sup>31</sup>

## 7 Monetary and Fiscal Policy

We now turn to consideration of monetary and fiscal policy in a secular stagnation. As we show in this section, the policy implications of our model for both monetary and fiscal policy are starkly different from existing New Keynesian models of the zero lower bound.

Let us first consider the effect of an increase in the inflation target  $\Pi^*$ . This change has no effect on the AS curve but instead shifts the AD curve. Specifically, a rise in the inflation target shifts out the kink point in the AD curve as shown in the left-hand panel of Figure 6. In this figure, the initial inflation target is set at one percent. As the inflation target increases, the kink point moves upward, effectively shifting up the downward sloping portion of the AD curve.  $AD_1$  shows the original aggregate demand curve with a unique secular stagnation steady state.  $AD_2$  shows the effect of a modest increase in the inflation target, while  $AD_3$  shows the effect of a large increase in the inflation target. Notice that  $AD_3$  now intersects the aggregate supply curve at three points.

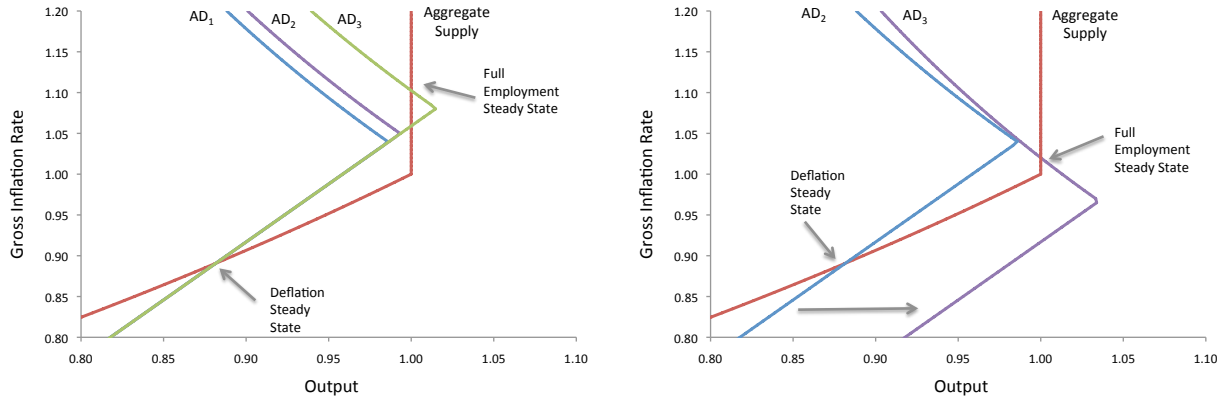
$AD_2$  illustrates the perils of too small an increase in the inflation target. For a sufficiently negative natural rate of interest, a small increase in the inflation target will not shift the AD curve enough to intersect the full employment line. The contention that a small increase in the inflation target will be ineffective has been labeled by Krugman variously as the “timidity trap” or, in reference to Japan in the late 1990s, as the “law of the excluded middle.” Our framework readily captures this idea. Formally, the inflation target needs to be high enough so that  $(1 + r^f) \Pi^* \geq 1$ ;

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mechanisms need not work solely through the labor market - see Garga and Singh (2016) for a model with hysteresis effects on productivity growth.

<sup>31</sup>To capture the slowdown in GDP per capita growth and the absence of outright deflation in the US and Eurozone, we extend our basic model to incorporate hysteresis effects on productivity growth and a more general wage norm where nominal wages are indexed to productivity growth and the inflation target. See Appendix G.

Figure 6: Monetary and fiscal policy responses



otherwise, the kink point in the AD curve occurs to the left of the full employment line.

With a sufficiently large increase in the inflation rate (as shown by  $AD_3$ ), our model admits three distinct steady states. The first steady state at the top intersection of the two curves is the normal, full-employment equilibrium at which point inflation is equal to the inflation target of the central bank  $\Pi^*$ . At this point the nominal interest rate is positive because the inflation target is large enough to accommodate the negative natural rate - that is,  $(1 + r^f)\Pi^* > 1$ . However, there is another equilibrium at full employment which is consistent with the policy rule. This is the second intersection of the two curves where  $i = 0$  and  $\Pi < \Pi^*$ . This steady state, however, is locally indeterminate.<sup>32</sup> Importantly, even with a sufficiently large increase in the inflation target, the secular stagnation steady state remains; an increase in the inflation target does not eliminate this equilibrium. Furthermore, this steady is locally determinate.

This multiplicity shows that monetary policy is less effective in our environment than in models that feature temporary liquidity traps such as [Krugman \(1998\)](#) or [Eggertsson and Woodford \(2003\)](#). In those models, a permanent increase in the inflation target will always have an effect because, by assumption, one can always reach the higher inflation target at some point in the future. Working backwards, a commitment of this sort will always have expansionary effects during a liquidity trap, and, provided the inflation target is high enough, it may even eliminate the demand slump altogether. Since the trap is permanent in our model, however, this backward induction breaks down; there is no future date at which one can be certain that the higher inflation target is reached (even if the policy regime is fully credible in the sense that people do not expect the government to deviate from the policy rule). For the same reason, a commitment to keep nominal rates low for a long period of time in a secular stagnation is of limited use and does not by itself guarantee a recovery. Indeed, interest rate commitments of the type pursued by the Federal Reserve during the crisis (often referred to as "forward guidance") would be irrelevant in shifting

<sup>32</sup>This steady state is akin to the deflation steady state in [Benhabib, Schmitt-Grohé and Uribe \(2001\)](#).

the economy out of the deflationary equilibrium since households are expecting rates to stay at zero forever. Even if a recovery were anticipated, the expansionary effects of a commitment to keep interest rates lower in the future is far less effective, given discounting in the Euler equation due to finite lifetimes.<sup>33</sup> Though an increase in the inflation target could make a full-employment steady state feasible, our model is silent on how a government could coordinate expectations on the favored full-employment equilibrium.

The fact that we have two locally determinate steady states suggests that there is no obvious reason why a higher inflation target should bring about full employment for an economy at the secular stagnation steady state.<sup>34</sup> Moreover, as shown by [Gibbs \(2017\)](#), both the full employment and secular stagnation steady state are learnable and survive under deviations from rational expectations.<sup>35</sup>

Given the drawbacks of monetary policy, we turn to fiscal policy. We extend our model to incorporate taxes and denote taxes on each generation by  $T_t^i$  where  $i = y, m$  or  $o$ . We first consider the effect of fiscal policy on the natural rate (i.e. in the endowment economy) before reincorporating nominal frictions. The budget constraints can now be written taking taxes into account:

$$C_t^y + T_t^y = B_t^y \quad (31)$$

$$C_{t+1}^m + (1 + r_t) B_t^y = Y_{t+1}^m - T_{t+1}^m - B_{t+1}^m \quad (32)$$

$$C_{t+2}^o = Y_{t+2}^o + (1 + r_{t+1}) B_{t+1}^m - T_{t+2}^o \quad (33)$$

Public debt now enters the asset market clearing condition:

$$-N_{t-1}B_t^m = N_tB_t^y + N_{t-1}B_t^g \quad (34)$$

where  $B_t^g$  is government debt (normalized in terms of the size of the middle generation). In previous sections, the only demand for borrowing came from the young households; now the government also may want to borrow so that loan demand is given by the right-hand side of equation

<sup>33</sup>See [McKay, Nakamura and Steinsson \(2016\)](#) for a discussion.

<sup>34</sup>One suggestion, proposed by John Cochrane in discussing this paper, is that the government can select between the two determinate steady states by raising nominal interest rates since with a high enough inflation target the nominal interest rate is positive. The main problem with this interpretation is that it presumes the Taylor rule is a structural policy regime that can be changed by increasing the nominal rate, so that increasing interest rates today will necessarily imply an increase in nominal interest rate forever. If one instead interprets the Taylor rule as a reduced form representation of a policy regime that aims at targeting a certain level of inflation  $\Pi^{ast}$ , then an increase in nominal interest rate at time  $t$  need not imply anything about future policy commitment. If an increase in the nominal interest rate at time  $t$  only changes the interest rate at that time, without affecting expectations, then it will have a contractionary effect via the aggregate demand channel. This could be modeled more formally as optimal policy under discretion (Markov Perfect Equilibrium), for example, whereby the policy objective of the government is to minimize the deviation of inflation from a target.

<sup>35</sup>Another consideration against raising the inflation target and accommodating a negative natural rate of interest are concerns that very low rates could spur asset price bubbles and raise financial stability concerns. See [Galí \(2014\)](#) and [Asriyan et al. \(2016\)](#) for further discussion.

(34):

$$L^d = \frac{1+g}{1+r}D + B^g \quad (35)$$

We have omitted time subscript to indicate that we are evaluating the steady state. Meanwhile, the supply of loans can be derived in exactly the same way as before yielding:

$$L^s = -B^m = \frac{\beta}{1+\beta} (Y^m - D - T^m) - \frac{1}{1+\beta} \frac{Y^o - T^o}{1+r} \quad (36)$$

Relative to the earlier specification, loan supply must now also track tax payments. The government's budget constraint is the final equation needed to determine asset market equilibrium:

$$T^m + B^g + \frac{1}{1+g}T^o + (1+g)T^y = G + (1+r)\frac{1}{1+g}B^g \quad (37)$$

where  $G$  is government spending (normalized in terms of the size of the middle generation). The equilibrium real interest rate is once again the interest rate that equalizes loan supply  $L^s$  and loan demand  $L^d$  and takes the same form as we saw previously in Section 3 and illustrated by Figure 2. The key difference is that now the real interest rate can be affected by fiscal policy, shifting the loan supply and loan demand curves in Figure 2. A fiscal policy regime corresponds to a choice of the level and distribution of taxation and government spending  $(T^o, T^m, T^y, G, B^g)$  subject to the government's budget constraint. The overall effect of fiscal policy on the real interest rate then depends on the contribution of all the fiscal variables. Equating (35) and (36), and taking account of the budget constraint (37), we have two equations and six unknown variables  $(T^o, T^m, T^y, G, B^g, 1+r)$ . Hence, we need four restrictions on the fiscal policy instruments to determine the interest rate. Previously, we implicitly assumed that  $T^o = T^m = T^y = 0$ ,  $G = 0$  which implies  $B^g = 0$  from the government budget constraint, leaving equations (35)-(36) to pin down the real interest rate  $1+r$ .

Consider now a more general policy regime that will help us clarify a number of results.<sup>36</sup> The tax on the young and government spending are exogenously given by  $T^y = T^*$  and  $G = G^*$ . Similarly, the overall level of *real* government debt is exogenously given by  $B^g_*$ . Finally, we assume that taxes on the middle aged and the old satisfy the following relation:

$$T^m = \frac{1}{\beta} \frac{1}{1+r} T^o = T \quad (38)$$

where the level of taxation  $T$  adjusts so that the budget constraint (37) is satisfied. The distribution of taxation condition given by (38) ensures that now there is no effect of fiscal policy on the supply of loans given by (36).

We first consider an increase in public debt  $B^g_*$ . Under the fiscal rule we have considered, an increase in public debt only affects loan demand, shifting out the demand for debt and raising the

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<sup>36</sup>While this particular policy regime is clearly special, we think it helps illustrate relatively cleanly how fiscal policy operates in this environment.

natural rate. In this respect, increasing government debt is a natural way of avoiding a secular stagnation.<sup>37</sup> Who receives the proceeds from this increase in government debt? The conditions above show that this does not matter so long as equation (38) is satisfied. The increase in government debt could be directed to the young, towards government spending, or distributed to the middle aged and the old in accordance with the fiscal rule (38). The effect of public debt also shows that if a central bank kept the *nominal* level of money constant in a secular stagnation with deflation, eventually, the natural rate would have to rise (see Appendix F for more details). We can similarly explore the effect of increasing government spending, funded via taxes, or various tax redistribution schemes. The effect of those policies can be gauged by analyzing how the policy shifts loan demand and loan supply in Figure 2. We will be a bit more specific about these type of policy experiments in the full lifecycle model.

A critical reason for why an increase in government debt raised the natural rate of interest is that it is expected to be permanent. To make this clear, consider the following policy regime.  $G_t = T_t^y = B_{t-1}^g = 0$  and:

$$\begin{aligned} T_t^m &= -B_t^g \\ T_{t+1}^o &= (1 + r_t) B_t^g \end{aligned}$$

The thought experiment here is that an increase in the public debt results in a lump sum transfer to the middle aged in period  $t$ . The old are then taxed by the same amount in the next period (plus interest) to return the debt to its original level.<sup>38</sup> A temporary increase in the public debt is completely irrelevant for the interest rate: the increase in debt in period  $t$  is met by a increase of the middle aged in their own savings to pay off the future tax. The point is that the effect of an increase in the public debt considered in the previous policy regime depends critically on agent's expectations about future fiscal policy. In particular, it depends upon the expectation of the middle aged that they will not be taxed to pay down the debt in the future.<sup>39</sup>

So far, we have only considered the effect of fiscal policy on the real interest rate in the endowment economy. Our results, however, carry over to the full model with endogenous production. Fiscal policy that leads to a change in the real interest rate in the endowment economy is equivalent to a policy that changes the natural rate of interest in the model with production. It therefore corresponds directly to a shock that shifts the AD curve, as shown in right-hand panel of Figure 6, displaying the effect of an increase in government spending via debt issuance. Thus, fiscal policies that raise the natural rate of interest correspond to an outward shift in the aggregate demand

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<sup>37</sup>The ability of an increase in the public debt in an OLG economy to undo the effect of credit frictions (that is, the effect of borrowing constraints) is similar to examples presented in Woodford (1990).

<sup>38</sup>For simplicity, we set population growth to zero.

<sup>39</sup>Since money and public debt are perfect substitutes at the zero lower bound, a *temporary* helicopter drop would also have the same effect. A helicopter drop is only effective to the extent that the rise in consolidated government liabilities is permanent.

**Table 1:** Government purchases multiplier at zero lower bound

Financing	Multiplier	Value
Increase in public debt	$\frac{1+\beta}{\beta} \frac{1}{1-\kappa\psi}$	$> 2$
Tax on young generation	0	0
Tax on middle generation	$\frac{1}{1-\kappa\psi}$	$> 1$
Tax on old generation	$-\frac{1+g}{\beta} \frac{1}{1-\kappa\psi}$	$< 0$

curve, while policies that reduce the natural rate correspond to an inward shift of the AD curve. Importantly, in contrast to monetary policy where an increase in the inflation target only allowed for the possibility of a “good equilibria” and suffered from multiple steady states, fiscal policy can eliminate the secular stagnation equilibrium since it shifts the entire AD curve (see Figure 6).

To derive some analytic results for the effect of fiscal policy on output, let us generalize equation (29) by combining equations (35) and (36), together with equations (16), (25) and assuming  $i = 0$ , to yield:

$$Y = D + T^m + \frac{1+\beta}{\beta} B^g + \left( \frac{(1+g)(1+\beta)}{\beta} D - \frac{1}{\beta} T^o \right) \Pi \quad (39)$$

where we now see how fiscal instruments directly affect aggregate demand at a zero interest rate. Tracing out exactly how aggregate demand shifts requires being specific about the policy regime. In Table 1, we derive analytically the steady state multiplier of government spending at the zero lower bound under different financing conditions, under the policy regime we specified before (except we relax equation (38) when government spending is financed by the middle aged or old).<sup>40</sup>

Observe that, away from the ZLB, the multiplier is zero since labor is supplied inelastically; once all workers are employed, government purchases will reduce private consumption one to one without any effect on output. At the zero bound, however, the multiplier is generally different from 0 as seen in Table 1 and critically depends on way in which government spending is financed.<sup>41</sup>

First, we consider the case in which spending is financed via an issuance of public debt. Financing via an increase in the public debt results in the largest multiplier shown in Table 1. Using

<sup>40</sup>The parameters  $\kappa$  and  $\psi$  - the slope of the AS and AD curves respectively - are given by:

$$\kappa = \frac{1-\alpha}{\alpha} \frac{1-\gamma}{\gamma}$$

$$\psi = \frac{1+\beta}{\beta} (1+g) D$$

<sup>41</sup>In computing the multiplier, we consider only a small increase in spending (so the zero bound is still binding) and a linear approximation of the model around a zero inflation steady state to compute this statistic.



the formula in Table 1, we see that because  $\beta$  is less than one, and  $\kappa$  can be no lower than zero (when wages are perfectly fixed), this multiplier has to be larger than 2. As we increase the value of  $\kappa$ , the multiplier becomes larger and can even be unboundedly large - a similar result as found in [Christiano, Eichenbaum and Rebelo \(2011\)](#) in the context of the standard New Keynesian model.

However, in contrast to the NK model, government spending multipliers are not always positive at the ZLB. The sign of the multiplier depends on whether fiscal expansions reduce the saving glut in a secular stagnation, which, in turn, depends on how government spending is financed. If instead of being financed by increasing debt, spending is financed via a tax on young households, the multiplier is zero. The collateral constraint is binding in equilibrium so that the young will cut their consumption by exactly the same amount as they are taxed. An increase in government purchases then simply substitutes for existing consumption by the young leaving output unchanged. If an increase in spending is financed via a tax on middle-generation households, the purchases multiplier is still positive but smaller than if financed via debt as shown in Table 1. We see from the analytic expression that this multiplier always has to be greater than one. Finally, if government purchases are financed via a tax on the old, the multiplier is negative. This negative multiplier obtains because the old will cut their spending one for one with the tax (thus offsetting the higher spending by the government). Meanwhile, the middle age will now increase their saving due to anticipation of higher future taxes which reduces aggregate demand.

Nevertheless, overall, the model suggests a relatively positive picture of fiscal policy as it can lead to an increase in demand via either debt policy or tax and spending actions.

## 8 A Quantitative Lifecycle Model

We now turn to a medium scale quantitative version of the baseline model where we can incorporate a variety of additional features.<sup>42</sup> The economy consists of a large number of households with identical utility functions. Households enter economic maturity at age 26, after which they work, consume, have children, and trade in asset markets. Households pass away with certainty at age  $J$ , which we take to be 81 years. Households have children at age 26, and the population growth rate is determined by the total fertility rate ( $\Gamma$ ) of every family. Households face a probability of dying stochastically before reaching maximum age  $J$ . The probability of surviving between age  $j$  and  $j + 1$  is denoted by  $s_j$ .<sup>43</sup> The unconditional probability of reaching age  $j$  is denoted with a superscript  $s^j$ .<sup>44</sup>

Households receive utility from two sources: (1) consumption, which is given by a time-

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<sup>42</sup>The model is based on [Auerbach and Kotlikoff \(1987\)](#) and [Ríos-Rull \(1996\)](#).

<sup>43</sup>Age specific survival rates may also vary over time  $t$ , however for notational simplicity we omit these additional subscripts.

<sup>44</sup>This can be calculated as the product of the one-period survival probabilities:  $s^j = \prod_{m=26}^j s_m$ .

separable, constant elasticity of substitution (CES) utility function  $u(\cdot)$  with elasticity parameter  $\rho$ , and (2) bequests that are divided equally among all descendants. The bequest motive is also characterized by a CES function  $v(\cdot)$  whose argument is the amount of bequests left *per descendant*, denoted by  $x$ .<sup>45</sup> The utility function for bequests is multiplied by a parameter  $\mu \geq 0$  that determines the strength of the bequest motive. Denoting consumption of household of age  $j$  at time  $t$  by  $c_{j,t}$  and the discount rate by  $\beta$ , a household that enters economic maturity at time  $t$  has lifetime expected utility given by the expression:

$$U_t = \sum_{j=26}^J s^j \beta^j u(c_{j,t+j-1}) + s^J \beta^J \mu v(x_{J,t+J-1})$$

A household of age  $j$  can trade in a real asset  $a_{j,t}$  at time  $t$  which is used as productive capital. At time  $t+1$ , capital will pay a return  $r_{t+1}^k$  which is the rental rate of capital, and has a resell value (net of depreciation) of  $\xi_{t+1}(1-\delta)$ , where  $\xi_{t+1}$  is the exogenous relative price of capital in terms of the consumption good. Each household has an identical exogenous labor productivity process, or human capital profile, denoted by  $hc_j$ , which changes with age. Households receive no wage income after retirement, which, in our model, occurs after age 65. We assume an inelastic labor supply, hence wage income is equal to the wage multiplied by the household's age-specific labor productivity  $hc_j$  net of labor taxes  $(1-\tau^w)$ .

Households also receive income from the pure profits of firms, denoted by  $\pi_{j,t}^f$ , and we assume that profits are distributed proportionally to labor income. Finally, the household may receive a bequest  $q_{j,t}$ . Individuals receive bequests  $q$  the year after their parents die, thus descendants receive bequests at age  $J-24$ . For example, if we take  $J$  to be 81, individuals would receive bequests at age 57. Bequests made ( $x$ ) are zero at all times except in the final year of life at age  $J$ . Following [Ríos-Rull \(1996\)](#), we suppose that agents insure themselves against the idiosyncratic risk of early death via one-period annuity contracts.

The flow budget constraint of a household of age  $j$  at time  $t$  is:

$$c_{j,t} + \xi_t a_{j+1,t+1} + \Gamma_{26,t-j+26} \cdot x_{j,t} = (1-\tau^w)w_t hc_j + \pi_{j,t}^f + \left(r_t^k + \xi_t(1-\delta)\right) \left(a_{j,t} + q_{j,t} + \frac{1-s_j}{s_j} a_{j,t}\right)$$

Households can borrow against future income, and we impose a borrowing constraint of the same form as in our simple model of Section 3.<sup>46</sup>

$$a_{j,t} \geq \frac{D_t}{1+r_t}$$

There are two types of firms: final and intermediate goods producers. The final goods firms

<sup>45</sup>Thus the total size of the bequest left by households is the bequest  $x$  multiplied by the fertility of the household.

<sup>46</sup>We will assume, in addition, that the borrowing constraint  $D_t$  grows at the rate of productivity growth and the household's earning potential over the life cycle.

produce a differentiated good  $Y_t^f$ . The final good composite is the CES aggregate:

$$Y_t = \left( \int_0^1 \left( Y_t^f \right)^{\frac{\theta_t-1}{\theta_t}} df \right)^{\frac{\theta_t}{\theta_t-1}}$$

Each final good producer utilizes  $Y_t^m$  of intermediate good to produce output via a linear technology:  $Y_t^f = Y_t^m$ . The presence of monopolistically competitive final goods firms allows for a time-varying markup given by  $\frac{\theta_t}{\theta_t-1}$ ; pure profits due to monopoly rents are returned to the households.

There is a perfectly competitive intermediate goods sector that sells their production to the final goods sector. These firms hire workers at wage rate  $w_t$  and rent capital at rate  $r_t^k$ . They operate a CES production function in labor and capital with an elasticity of substitution  $\sigma$ . The production function is:

$$Y_t^m = \left( \alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha) (A_t L_t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

where  $A_t$  is exogenous, labor augmenting technological progress, and  $0 < \alpha < 1$ .

The government purchases some level of output  $G_t$  and may issue debt. The budget constraint for the government becomes:

$$b_{g,t} = G_t + (1 + r_t) b_{g,t-1} - T_t$$

where taxes are collected on labor income. We economize on notation by omitting real and nominal bonds as assets above; these assets enter in the same way as in the simpler model (Section 5) so that there is both a well defined real interest rate  $r_t$  on a risk-free one period real bond and a nominal interest rate  $i_t$  on nominal bonds.

Monetary policy and wage rigidity are modeled as in Section 5. The full nonlinear model is solved numerically; we solve for both the stationary equilibrium and perfect foresight transition paths. We outline the computational details and the numerical solution algorithm in Appendix J.<sup>47</sup>

## 8.1 Calibration

We first calibrate our model to match the US economy in 2015. There remains considerable uncertainty about the size of the output gap in 2015. We consider two polar cases: a zero output gap in 2015 based on [Stock and Watson \(2012\)](#) and an alternative calibration where the output gap is  $-15\%$ ; this is what [Hall \(2016\)](#) estimates as the deviation of output from its pre-recession trend

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<sup>47</sup>The equilibrium conditions consist of roughly 170 equations in the stationary equilibrium. Transition dynamics are more complicated as the birthrate over the past 25 years becomes a state variable in that case and agents form expectations over the entire transition path.

**Table 2:** Parameters taken from the data and related literature

<i>Panel A: Data</i>	<i>Symbol</i>	<i>Value</i>	<i>Source</i>
Mortality profile	$s_{j,t}$		US mortality tables, CDC
Income profile	$hc_j$		Gourinchas and Parker (2002)
Total fertility rate	$n$	1.88	UN fertility data
Productivity growth	$g$	0.65%	Fernald (2012)
Government spending (% of GDP)	$G$	21.3%	CEA
Public debt (% of GDP)	$b_g$	118%	Flow of Funds
<i>Panel B: Related literature</i>			
Elasticity of intertemporal substitution	$\rho$	0.75	Gourinchas and Parker (2002)
Capital/labor elasticity of substitution	$\sigma$	0.6	Antras (2004)
Depreciation rate	$\delta$	12%	Jorgenson (1996)

in 2015. We consider this latter case as a sensible upper bound on the possible size of the output gap.<sup>48</sup>

We first consider the case of zero output gap. We calibrate our model to match the 2015 real interest rate in the US, which was  $-1.47\%$ . Since we are assuming zero output gap, this implies that the natural rate of interest is also  $-1.47\%$ . We will thus be on the upper section of the aggregate demand and aggregate supply curves, and can continue with our parameterization without reference to nominal frictions.

Our parameters come from three main sources. The first is statistical data about US demographics and the economy that we can match directly, such as mortality rates, fertility levels, productivity growth, and the size of government debt. The second source of parameters we take directly from the related literature which has previously estimated or calibrated these parameters. The third set are chosen to match key moments in the data, such as the investment to output ratio. The moments will be matched by minimizing an objective function. We discuss each in turn.

Panel A of Table 2 shows the first category of parameters that are taken directly from observed data. We use mortality data from the Centers for Disease Control (CDC) to directly match US survival tables. The total fertility rate is taken from UN fertility data, and the retirement length is chosen to match the average years of retirement. Government debt to GDP and government spending to GDP are also chosen to match current values.<sup>49</sup> The rate of productivity growth is a

<sup>48</sup>To be clear, Hall (2016) does not interpret this gap as reflecting output gap of the kind we see in the model, but we still think that computing deviation from trend provides a convenient benchmark to contrast with no output gap at the opposite extreme.

<sup>49</sup>We set government debt as the sum of federal and state and local debt as reported by the Council of Economic Advisors and the Census Bureau.

**Table 3:** Parameters chosen to match targets

<i>Targets</i>	<i>Model/Data</i>	<i>Source</i>
Natural rate of interest	−1.47%	Federal Reserve
Investment to output ratio	15.9%	NIPA
Consumer debt to output ratio	6.3%	Flow of Funds
Labor share	66.0%	Elsby (2013)
Bequests to output	3.0%	Hendricks (2001)

<i>Parameters chosen to match targets</i>	<i>Symbol</i>	<i>Value</i>
Rate of time preference	$\beta$	0.98
Borrowing limit (% of annual income)	$D$	23.4%
Bequests parameter	$\mu$	21.6
Retailer elasticity of substitution	$\theta$	4.9
Capital share parameter	$\alpha$	0.24

key determinant of the real interest rate. Our baseline uses a productivity growth rate of 0.65% per year which we have taken from [Fernald \(2012\)](#). The wage profile  $hc_j$  is chosen to match the earnings profile estimated from the data by [Gourinchas and Parker \(2002\)](#).

Panel B of Table 2 shows the second category of parameters taken from the related literature. We set the intertemporal elasticity of substitution  $\rho$  equal to 0.75. This parameter has been estimated widely in the literature, with ranges between .25 and 1.<sup>50</sup> The depreciation rate comes from [Jorgenson \(1996\)](#) and [BEA \(2004\)](#), who have extensive estimates of the depreciation rate of private and governmental nonresidential equipment.<sup>51</sup> The value of the production elasticity of substitution,  $\sigma$ , has also been estimated widely in the literature, and generally falls between .4 and 1.<sup>52</sup> As these parameters are not directly observed in the data, we report in Appendix L how our results change for different parameter choices and will also comment on them below.

We choose the remaining parameters to match five key moments of the data as of 2015: a real interest rate of −1.47%, an investment to output ratio of 15.9%, a consumer debt to output ratio of 6.3%, a labor share of 66.0%, and a bequest to output ratio of 3%.<sup>5354</sup> The parameters chosen this

<sup>50</sup>For example, [Cooley and Prescott \(1995\)](#) sets  $\rho = 1$ , while [Auerbach and Kotlikoff \(1987\)](#) and [Ríos-Rull \(1996\)](#) set  $\rho = .25$ .

<sup>51</sup>Since our model does not explicitly incorporate housing, our preferred specification uses depreciation applicable to equipment investment, which generally is higher than for housing. We consider a lower value in the sensitivity analysis in the Appendix.

<sup>52</sup>For example, [Antras \(2004\)](#) estimates it is between 0.4 and 0.9 while [Oberfield and Raval \(2014\)](#) estimate an elasticity of 0.7, and [Klump, McAdam and Willman \(2007\)](#) estimate a value between 0.5 and 0.6.

<sup>53</sup>Sources are listed in Table 3.

<sup>54</sup>We choose to target the investment to output ratio, rather than capital output ratio as this ratio does not require

way are the rate of time preference  $\beta$ , the debt limit  $D$ , the bequest parameter  $\mu$ , the capital share parameter  $\alpha$ , and the retailer elasticity of substitution  $\theta$ . We select the parameters by minimizing a loss function whose value is the sum of the squared differences between the moments of the model and those we take from the data.

Generically, there is no one-to-one mapping between these remaining parameters and the targets. Hence, we jointly choose all parameters to match the model output to the targets. Nevertheless, each of the parameters above correspond relatively closely with one of the key moments we are trying to match. The rate of time preference  $\beta$  has a direct effect on the real interest rate; as  $\beta$  increases, the real interest rate falls. Not unexpectedly, the debt limit most directly affects the level of consumer debt to output, while the bequest parameter  $\mu$  mostly directly affects the bequest to output ratio. Finally, the capital share parameter  $\alpha$  determines the investment to output ratio, while inverse profit share  $\theta$  controls the labor share.

## 8.2 Negative Real Interest Rates

Table 3 shows the results of calibration procedure with zero output gap. With the parameters selected by our minimization, each of model moments perfectly matches those in the data. We document in Appendix N that the parameters in Table 3 fall directly within the range of parameter values reported in the existing literature. Therefore, we argue that a relatively standard OLG model with capital accumulation calibrated to match US data, can generate permanently negative real interest with standard parameter values.<sup>55</sup>

The result that a standard, conservatively calibrated OLG model can deliver a substantially negative natural rate of interest suggests a reevaluation of the standard estimate of the natural rate for the US, and, indeed, that there is no reason a priori to expect a normalization of interest rates in the US. Pre-recession estimates for the long-run neutral real interest rate remained between 2% and 3% (see Williams (2016)). As long-term rates have continued to fall, the Federal Reserve has adjusted downward its estimates of the neutral rate to between 0.5% and 1.8% as of Federal Reserve Open Market FOMC (2016) (FOMC) projections. Our calibration, however, suggests that given current productivity and demographic trends, these estimates for the long-run neutral rate

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adjustment of the past capital stock for changes in the relative price of capital goods - the investment to output is just a ratio of two nominal, readily measured quantities.

<sup>55</sup>In the Appendix L, we show the optimal consumption path for households over their lifecycle. Consumption tracks income over the early part of the lifecycle due to borrowing constraints, then declines gradually in order to save for retirement. This leads to the classic “hump-shaped” consumption profile. For comparison purposes, this figure also includes the estimated consumption profile from the Consumer Expenditure Survey, as estimated by Gourinchas and Parker (2002). We also display the model’s population pyramid in Appendix L as compared to current US pyramid. The fact that we are considering the stationary equilibrium is not innocuous. It does not take into account the dynamics of the aging of the baby boom population and its effect on interest rates as the boom filters through the population pyramid. To study these effects, we need to consider transition dynamics, which we turn to shortly.

remain too optimistic. Our benchmark calibration has the steady state real interest rate at -1.47% - or 2.27% lower than the FOMC estimate as of 2016.

Our results are robust to a variety of additional specifications; in particular, we focus on three parameters taken from the literature. Appendix L calculates the same moments in Table 3 for three alternate specifications: (1) setting depreciation to 8%, (2) setting the intertemporal elasticity of substitution  $\rho$  to 1, (3) setting the production elasticity parameter  $\sigma$  to 1 (Cobb-Douglas). For each of these specifications, we again minimize the objective function to find  $\beta$ ,  $\alpha$ ,  $\mu$ ,  $D$ , and  $\theta$  to match the targets from the data. The results for these alternate specifications are similar to our main specification. The model still hits the targets, but with different values for the calibration parameters that still remain well within the ranges from the literature. We report results under these alternative parameterizations for other experiments in the remainder of this section in Appendix L.

While the model can clearly generate permanently negative real interest rates, it is also of interest to explore if it can explain the reduction observed in real interest rate over time. Before getting there, however it is worth asking if our model can also replicate a scenario in which the natural rate of interest is negative enough that the zero bound is binding with output below potential. This is what we turn to next.

### 8.3 Secular Stagnation in a Quantitative Lifecycle Model

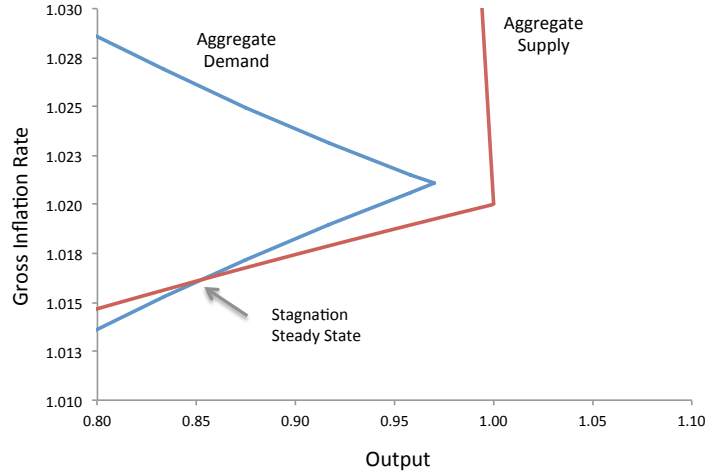
We keep the first group of parameters (those taken directly from the data) and the second group of parameters (those taken from the literature) the same as in our initial calibration. For our third group of parameters (those we choose to match moments), we keep all parameters unchanged save for the rate of time preference  $\beta$ . In addition, we must choose the wage rigidity parameter  $\gamma$ .

We jointly choose  $\beta$  and  $\gamma$  to match two moments of the US economy in 2015: the inflation rate and the output gap. We model the economy as in a secular stagnation, with an inflation rate of 1.62% as in the data and an output gap of -15%, corresponding to the deviation of output from its pre-crisis trend in 2015 as documented in Hall (2016). A value of  $\gamma$  of .91 and a discount rate of  $\beta$  of 0.99 match these moments. Our calibrated value of  $\gamma$  falls within the range wage rigidity estimated by Schmitt-Grohé and Uribe (2016). The value  $\beta$  is also consistent with common estimates from the OLG literature (see Appendix N).

Figure 7 shows the aggregate demand and aggregate supply curves of the secular stagnation stationary equilibrium. The two curves cross at an output gap of -15%. The natural interest rate in this calibration is -2.2% and therefore an inflation target of 2% is too low to allow the nominal interest rate to remain above zero. As a result, the economy is drawn into the secular stagnation steady state. The equilibrium real interest is -1.62% and the inflation rate is 1.62% - 40 basis points below target. Investment to output, bequests to output, labor share, and consumer debt to output



**Figure 7: Secular stagnation equilibrium**



continue to closely match the US moments.<sup>56</sup>

Figure 7 represents the second key quantitative finding of the paper. It suggests that the model can replicate a permanent stagnation with a standard parameterization of precisely the same form as we have shown in the simple model. As in the 3-period model, a surplus of savings over investment opportunities pushes the equilibrium interest rate negative. If the inflation target is insufficient to prevent the zero lower bound from binding and nominal wage rigidities bind, a secular stagnation emerges.

#### 8.4 Decline in Interest Rates since 1970

We use our model to quantitatively analyze the fall in the real interest rate observed since 1970 and assess the relative contribution of different factors. 1970 is a natural starting point, for it is the point at which a number of economic trends began shifting. Briefly, they are: (1) children of the baby boom generation (born beginning in 1945) begin to enter the labor force (2) mortality rapidly falls with life expectancy increasing from an average of 70.8 in 1970 to 78.7 in 2010 (3) according to [Gordon \(2016\)](#), the fall in the rate of productivity growth began in the 1970s (4) [Greenwood, Hercowitz and Krusell \(1997\)](#) and [Fernald \(2012\)](#) have documented that the relative price of investment goods has fallen by 30% since 1970 (5) [Karabarbounis and Neiman \(2014\)](#) have documented a significant drop in the labor share since 1970. Also, this period saw a significant rise in government and personal debt. For a full description of the changes in the economy since 1970, see Appendix M. For the data series used for these variables, see the Data Appendix.

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<sup>56</sup>The investment to output ratio, consumer to debt ratio, labor share and bequest to output remain very close to their value in Table 3, hence we choose not to reparameterize the model for  $\mu$ ,  $\alpha$ ,  $D$ , and  $\theta$  relative to the benchmark calibration.

**Table 4:** Change in parameters from 1970 to 2015

<i>Panel A: Data</i>		
	1970	2015
Life expectancy	70.7	78.7
Total fertility rate	2.8	1.9
Productivity growth	2.02%	0.65%
Government debt (% of GDP)	42%	118%
<i>Panel B: Relative Price of Investment</i>		
Relative price of investment (index 100=2015)	130	100
<i>Panel C: Change in targets</i>		
Consumer debt to output ratio	4.2%	6.3%
Labor share	72.4%	66.0%

We begin by targeting our calibration of the US economy to 1970 as shown in Table 4. All type 1 parameters (those taken directly from the data) we adjust to their 1970 counterparts (see Panel A) with the exception of the lifecycle income profile which remains unchanged. All type 2 parameters (those taken from the literature) are kept the same with the exception of the relative price of investment goods (Panel B). For the type 3 parameters (those we take to match moments), we keep the utility and production parameters ( $\beta$ ,  $\alpha$ ,  $\mu$ ) unchanged. We adjust the collateral constraint  $D$ , to match the lower consumer debt to output ratio observed in 1970, and the profit share  $\theta$  to match the higher labor share observed in 1970s (Panel C). The 1970 values for these two parameters are then  $D=0.144$ , and  $\theta=7.93$ .

Table 5 shows the results of the 1970 simulation. The simulated 1970 real interest rate is 2.55%, extremely close to the 2.62% observed in the data. This is an important quantitative result; our model is able to match the entire size of the decline in real interest rates over the past 45 years. The model predicts an investment output ratio (which we did not explicitly target) slightly higher than what is observed in the data. By our choice of  $D$  and  $\theta$  the model matches the consumer debt to output ratio exactly as well as the labor share. Overall, the model does a reasonable job of explaining the fall in the real interest rate observed over the past 45 years by using observed changes in productivity, demographics, relative price of investment goods, credit constraints, and changes in the labor share.

Table 6 decomposes the contribution of each of these factors to the decline in interest rates. We change each parameter from its steady state value in 2015 to its steady state value in 1970, holding all other parameters constant. We then examine the effect of this change on the real interest rate. For example, changing productivity growth from its 2015 level of 0.65% per year to

**Table 5:** Simulation results for 1970

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	2.55%	2.62%
Investment to output ratio	19.0%	16.8%
Consumer debt to output ratio	4.2%	4.2%
Labor share	72.4%	72.4%

**Table 6:** Decomposition of decline in natural rate of interest: 1970-2015

Forcing variable	$\Delta$ in $r$	% of total $\Delta$
Total interest rate change	-4.02%	100%
Mortality rate	-1.82	43%
Total fertility rate	-1.84	43%
Productivity growth	-1.90	44%
Government debt (% of GDP)	+2.11	-49%
Labor share	-.52	12%
Relative price of investment goods	-0.44	10%
Change in debt limit	+.13	-3%

it's 1970 level of 2.02% results in an increase in the steady state real interest rate of 1.9 percentage points. The table shows a decomposition of the relative importance of all the other factors.<sup>57</sup> The reduction in fertility, mortality, and the rate of productivity growth play the largest role in the decrease in real interest rates. The main factor which has tended to counterbalance these forces is an increase in government debt. Changes in the labor share and the relative price of investment goods play a smaller role in explaining the decline in real interest rates. Similarly the increase in consumer debt limit does not have a significant effect on the evolution of the interest rate in this parameterization.<sup>58</sup>

<sup>57</sup>The decomposition is calculated by first adding up the total change in the real interest rate from all of these factors, and then dividing the effect of changing a particular forcing variable by the entire change. Numbers may not sum to unity due to interaction effects.

<sup>58</sup>As our calibration only takes account of consumer debt, we suspect that this force may play a substantially larger role once one takes account of housing purchases and the associated debt, along with firms borrowing and lending. Together these forces would have tended to increase the real interest rate in the past 45 years and lowered rates during the crisis. The increase in inequality observed during this period may have also been working to reduce interest rates; we have not taken account of in the quantitative section and leave for future research.

**Table 7:** Raising the natural rate of interest to 1%

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	3.28
Government debt (% of GDP)	118%	215%
Productivity growth	0.65%	2.43%
Relative price of investment goods	1.00	2.43

## 8.5 Raising the Natural Rate of Interest

If the natural rate of interest is indeed  $-1.47\%$  (the level of our baseline analysis), this poses a challenge for policymakers. With a 2% inflation target, small decreases in the natural rate will cause the zero lower bound to bind implying that downturns may be sharper and more persistent. We now consider an alternative thought experiment, taking as given that the natural rate is  $-1.47\%$  in 2015. We ask, what economic conditions would be needed to increase the steady state real interest rate to a positive territory of 1 percent? While this target is somewhat arbitrary, we find it to be a useful benchmark. With the Federal Reserve's inflation target of 2 percent, a natural rate of 1% would give policymakers a reasonable amount of room to respond to negative shocks that otherwise would lead to a binding ZLB. It also corresponds with the current expectations of the FOMC about long-run real interest rates. An alternative way of formulating our quantitative experiment is to ask which forcing variables in 2015 would one need to change to be consistent with the current FOMC projections.

Table 7 provides the results of this experiment, and shows that substantial changes in the underlying fundamentals are needed in order to increase the natural rate to 1%. Given current demographic trends, it is implausible that fertility will reverse its decline and dramatically increase to 3.28 births per woman. An increase in immigration could make up a portion of the difference. It would also be challenging to increase productivity growth to 2.43% per year given the headwinds to productivity noted by [Gordon \(2016\)](#) and that productivity growth has rarely exceeded 2% since 1970.

Of particular interest is the potential impact of an increase in government debt on interest rates, given that a higher public debt is an obvious policy lever. As Table 7 shows, government debt would need to double to roughly 215% of GDP to increase the natural rate to 1%. Such a large level of debt raises questions about the feasibility of this policy for we have not modeled any costs or limits on the government's ability to issue riskfree debt - an assumption that may be strained at such high levels. While these results suggest that there are several reforms that would tend to increase the natural rate of interest, the menu of options does not paint a particularly rosy picture relative to the alternative of raising the inflation target of the central bank.

## 8.6 Transition Dynamics

So far, we have confined our analysis to stationary equilibria. While this is one natural benchmark, we can also consider transition dynamics. This requires taking a strong stance on agents' expectations during the transition. Here, we document numerical experiments in which it is assumed that the economy was at a stationary equilibrium in 1970, and then project the model forwards, assuming agents have perfect foresight about the path of exogenous processes.

We feed into the model the dynamic paths for each of the forcing variables in Table 6, and calculate the resulting transition path. We document the time series for each forcing variable the Data Appendix. On impact in 1970, agents have perfect foresight about how each of the exogenous and endogenous variables. For example, in 1970 all agents presently alive will realize that there will be a productivity slowdown over the next 40 years and will adjust their optimal decisions accordingly. The forcing variables are set to their final steady state values after the year 2015.

Since agents enter the labor force at the age of 26, the birthrates from 1945 to 1970s are key state variables when that will affect the real interest rate beginning in the 1970s. Since individuals entering the labor force in 1970 (age 26) were born in 1945, we use as inputs to the model data on fertility from 1945 onwards to measure the size of each incoming generation. The birthrate data from 1945 are reflected below the solid line in the age pyramid for the US economy in Figure 9, but, in 1970, those generations are economically inactive, only entering the labor force gradually.

Figure 8 shows the full transition path for the real interest rate. The 1970 real interest rate is 2.55% and declines non-monotonically throughout the time period until it reaches  $-1.0\%$  in 2015. The interest rate exhibits a brief recovery in the mid to late 1990s as productivity growth increases, but this subsides by the mid 2000s. Note that the model projects that the interest rate will continue to decrease until it hits a nadir in 2020. After 2020, there are cycles in the real interest rate due to the echo effects of the baby boom. The economy gradually converges to the final steady state interest rate of  $-1.47\%$ .<sup>59</sup>

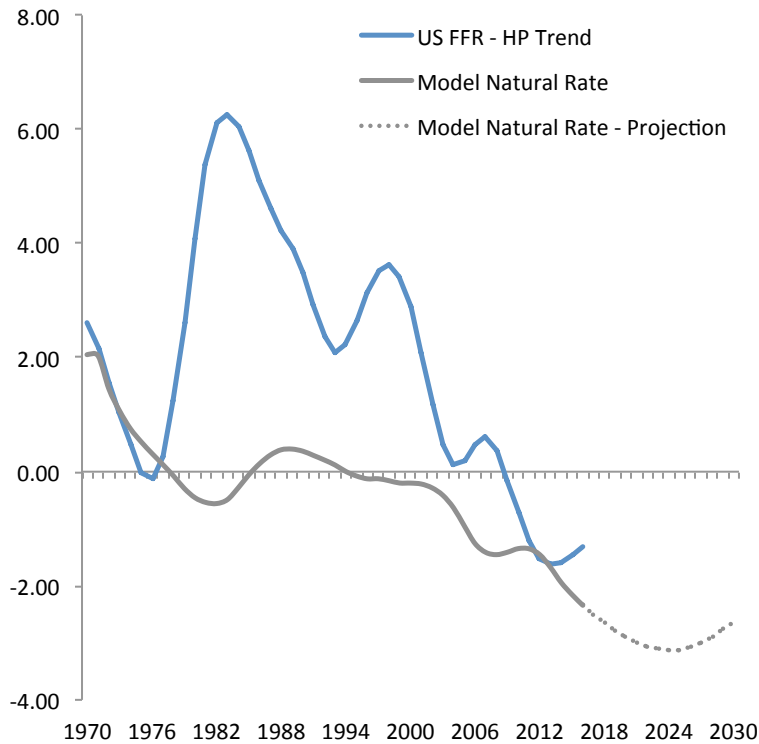
One interesting observation is that our model predicts a more rapid decline in the real interest rate than that observed in the data. A possible explanation for this, which we leave for future research, is offered by Summers (2014). He hypothesizes that the decline in the natural rate of interest rate since the late 1990s was masked by both the tech and housing bubble so that the true natural interest rate has only been observed post crisis. This could be modeled as exogenous fluctuations in the collateral constraint  $D$  in our model.

One interesting aspect of our model is that it does not generate an investment boom even though real interest rates are declining. In fact, investment actually falls in our model, from 19.0%

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<sup>59</sup>One factor we do not focus on in these transition paths is debt deleveraging. While the evidence and theory certainly suggests there was a significant deleveraging shock in 2008, our model as of yet does not include a housing sector. Since housing debt is a significant portion of consumer debt, we leave quantitative work of the effect of debt deleveraging on interest rates to future work.

**Figure 8:** Transition path of the natural rate of interest



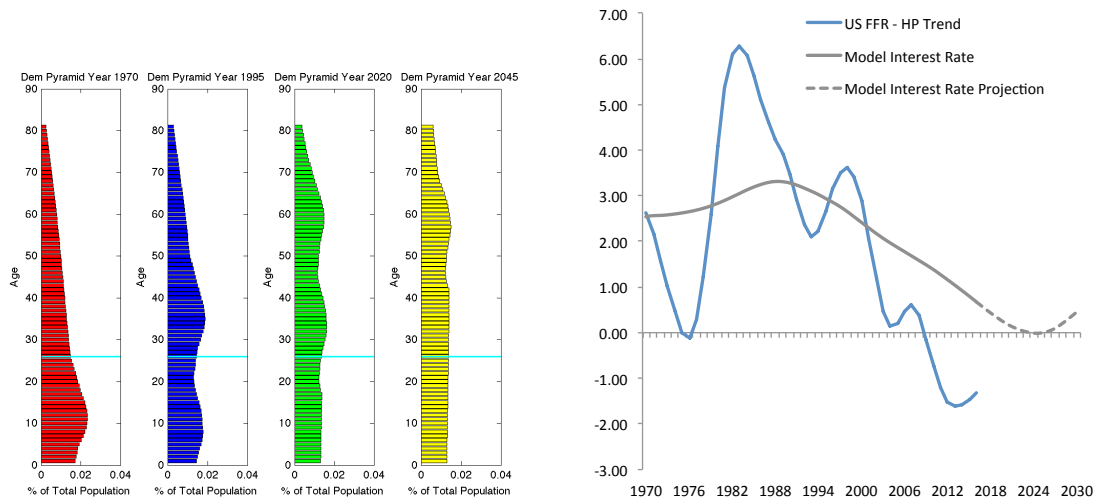
in 1970 to 15.9% in 2015. Although lower interest rates will tend to lead to a higher capital to output ratio and thus a higher investment rate, there are several counterbalancing forces in the model. In particular, a decrease in the rate of population growth and productivity growth lead directly to a lower investment to output ratio holding constant the capital to output ratio. Finally, our relatively inelastic production CES parameter means that the steady state  $K/Y$  ratio stays relatively constant for a given interest rate, which also dampens the effect on investment.

## 8.7 Effects of the Baby Boom

We now study the effects of the baby boom on real interest rates. As the baby boom started in 1945, the children of the baby boom did not enter economic maturity until they reached age 26 in 1970; thus, we are able to look at the impact of these demographic changes on the evolution of interest rates starting from our model economy in 1970.

The left-hand panel of Figure 9 shows the demographic pyramid for the model during the transition years. In 1970, the economically active population is assumed to be in steady state, thus the pyramid above the solid line is a triangle; the younger generation of the baby boom, not yet in the labor force, bulges the distribution, and will soon enter the labor force. Twenty five years later (1995), the bulge now shows up in the working age population. Fifty years later, in 2020,

Figure 9: Baby boom and bust



the baby boom generation is retired. There is also a second bulge in the population pyramid due to the children of the baby boomers (millennials). By the year 100, the population pyramids are approaching steady state.

The right-hand panel of Figure 9 shows the interest rate transition for the baby boom simulation. From 1970 until 2000, the baby boom tends to increase real interest rates due to an increase in the rate of population growth. After the baby boom comes the baby bust, and fertility rates drop as the boomers enter retirement. This leads to a sharp drop in real interest rates. The echo effects result in cycles in the real interest rate until it reaches its final steady state value of around 1%.

## 9 Conclusion

This paper builds a quantitative theory of negative interest rates and secular stagnation. We show how a low natural rate can lead to a secular stagnation: a persistent output slump, inflation below target, and a chronically binding ZLB. Our message is not that the ZLB will be binding forever with certainty. A world of low natural rates admits business cycles in which the short term rate can still be temporarily positive. Instead, it is a world characterized by “new normal,” in which real interest rates need on average to be negative to achieve full employment. As in the case of Hansen’s prediction in 1938, we hope that this gloomy outlook will prove to be wrong. If negative rates persist, our analysis has identified several measures that could eliminate secular stagnation via appropriate policy.

A serious challenge, however, is that our policy recommendations advocate in favor of policies that were considered vices rather than virtues in macroeconomic theory: a higher inflation target, persistent increases in the debt to GDP ratio, or even more generous pay as you go Social Security.



This poses a conundrum for policymakers who cannot know with certainty if we are indeed in a “new normal” or just in a prolonged period of low interest rates that will abate in the near future. If the “new normal” hypothesis is incorrect, then those very policies which are desirable to eliminate secular stagnation are likely to be counterproductive and costly as existing economic theory suggests. Given the uncertainty about the future, this creates new tradeoffs for optimal policy, which we have not touched upon but hope that future research considers.

In closing, we briefly discuss some of main arguments against the hypothesis of permanently negative interest rates, leaving a more detailed analysis to future research. The first argument, stated by Paul Samuelson and recently recapitulated by Ben Bernanke, is that negative real interest rates are a theoretical impossibility:

As ... Paul Samuelson taught me in graduate school at MIT, if the real interest rate were expected to be negative indefinitely, almost any investment is profitable. For example, at a negative (or even zero) interest rate, it would pay to level the Rocky Mountains to save even the small amount of fuel expended by trains and cars that currently must climb steep grades. It’s therefore questionable that the economy’s equilibrium real rate can really be negative for an extended period

Samuelson’s argument did not take into account two factors: (1) monopoly rents and (2) risk. In particular, under monopolistic competition, the rental rate of capital is given by  $r^k = \frac{1}{markup} \cdot MPK$ . With positive markups, the marginal product of capital is higher than the rental rate of capital. Thus, in equilibrium, there can be positive *social* returns to capital (even net of depreciation) while the rental rate (net of depreciation) and hence the real interest rate is negative. This is indeed the case in our quantitative model. Moreover, even though our calibrated model produces negative real interest rates, it can be shown that the economy is not dynamically inefficient.<sup>60</sup>

Adding risk to our model is another rejoinder to Samuelson’s argument. As [Abel et al. \(1989\)](#) show, adding aggregate risk can lead to a negative *riskfree* interest rate, while the average and marginal return from capital (net of depreciation) remains *positive*. While adding aggregate risk to our economy is beyond the scope of this current paper, we can illustrate this mechanism in a reduced form way. In [Appendix K](#), we extend the model by including an additional financial friction: all borrowing and lending must now occur through a bank, which charges a spread between the borrowing and lending rate of interest. With a spread of 5%, for example, it is possible to have a borrowing rate of interest which is positive while the lending rate of interest is negative. Since firms borrow at the higher rate, the marginal product of capital in excess of depreciation is positive in this economy.

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<sup>60</sup>The condition for dynamic inefficiency, following [Abel et al. \(1989\)](#), is that the economy spends more on investment in capital than it gets out of it in terms of production:  $I > MPK * K$

The second argument against the presence of secular stagnation conditions in the US is that that measured return on capital are stable (see, for example, [Gomme, Ravikumar and Rupert \(2015\)](#)). The argument is that the most relevant measure of return is not the return on government debt but the return on productive capital. One problem with that argument is business income does not just measure capital income, but it also measures pure profits.<sup>61</sup> And over the past decades, there is some evidence that competition has decreased and monopoly rents have risen.<sup>62</sup> If an increase in monopoly rents cancels out the decrease in competitive returns to capital, this would lead to a stable measured average return on capital.<sup>63</sup> This tends to increase the measured average return in our model through two channels: higher pure profits, and a higher marginal product of capital relative to the rental rate of capital. Although our economy exhibits a decline in the marginal product of capital over time, profits increase by enough to almost completely offset this decline, and thus the measured average return on capital stays constant. Future research will consider alternative measures of the return on capital and evidence from firm-level data on whether returns have fallen over time.

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<sup>61</sup>It is also possible that the NIPA accounts systematically understate the capital stock by mismeasuring intangible capital as discussed in [McGrattan and Prescott \(2010\)](#). To the extent that this bias has increased over time, the true return on capital may be declining as our model would suggest.

<sup>62</sup>See, for example, [Philippon and Gutierrez \(2016\)](#), [CEA \(2016\)](#), or [Decker et al. \(2016\)](#).

<sup>63</sup>An increase in monopoly rents is one possible explanation for the measured decrease in the labor share over the past 30 years. See in particular [Barkai \(2016\)](#), who finds an increase in markups is responsible for a decrease in both the labor and the capital share.

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# Online Appendix for

## *A Model of Secular Stagnation: Theory and Quantitative Evaluation*

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### A Derivation of Simple Model

#### A.1 Households' Problem

In this section, we specify and solve the household's problem in the general case of income received in all periods and taxes paid in all periods. For household  $i$ , their objective function and budget constraints are given below:

$$\max_{C_t(i), C_{t+1}(i), C_{t+2}(i)} \mathbb{E}_t \{ \log(C_t(i)) + \beta \log(C_{t+1}(i)) + \beta^2 \log(C_{t+2}(i)) \} \quad (\text{A.1})$$

$$\text{s.t.} \quad C_t(i) = w_t L_t(i) - T_t^y + B_t(i) \quad (\text{A.2})$$

$$C_{t+1}(i) = Z_t + w_{t+1} L_{t+1}(i) - T_{t+1}^m + B_{t+1}(i) - \frac{(1+i_t)}{\Pi_{t+1}} B_t(i) \quad (\text{A.3})$$

$$C_{t+2}(i) = w_{t+2} L_{t+2}(i) - T_{t+2}^o - \frac{(1+i_{t+1})}{\Pi_{t+2}} B_{t+1}(i) \quad (\text{A.4})$$

$$B_{t+j}(i) \leq \mathbb{E}_{t+j} (1 + r_{t+j+1}) D_{t+j} \quad \text{for } j = 0, 1 \quad (\text{A.5})$$

where the household  $i$  has exogenous labor supply endowments in each period of life,  $D_{t+j}$  is an exogenous collateral constraint, and  $T_{t+j}$  are lump sum taxes imposed by the government. We allow taxes to differ across household types and taxes to change over time.

We restrict ourselves to cases in which the collateral constraint is binding in the first period of life and possibly binding in the second period of life. In particular, we will assume two types of households - a household that has sufficiently low labor endowment in its middle period of life and remains credit constrained, and a household that has sufficiently high labor endowment in its middle period of life and is unconstrained. For the former, borrowing in the young and middle generations is determined by the binding collateral constraints. For the latter, borrowing is determined by the collateral constraint only while young; in the middle-generation, an Euler equation determines the optimal level of saving:

$$\frac{1}{C_t^{m,h}} = \beta \mathbb{E}_t \frac{1 + i_t}{\Pi_{t+1} C_{t+1}^{o,h}} \quad (\text{A.6})$$

Let  $L^y$  be the labor endowment for young generation,  $L^{m,l}$  be the labor endowment for the poor middle-generation household,  $L^{m,h}$  the labor endowment for the wealthy middle-generation household, and  $L^o$  the labor endowment in the last period. We adopt the normalization that

$L^y + \eta_s L^{m,l} + (1 - \eta_s) L^{m,h} + L^o = 1$ . The budget constraints for each type of household alive at any point in time is given below:

$$C_t^y = \alpha Y_t \frac{L^y}{L_t^{flex}} - T_t^y + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1 + i_t} \quad (\text{A.7})$$

$$C_t^{m,l} = \alpha Y_t \frac{L^{m,l}}{L_t^{flex}} + (1 - \alpha) Y_t - T_t^m - D_{t-1} + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1 + i_t} \quad (\text{A.8})$$

$$C_t^{m,h} = \alpha Y_t \frac{L^{m,h}}{L_t^{flex}} + (1 - \alpha) Y_t - T_t^m - D_{t-1} - B_t^{m,h} \quad (\text{A.9})$$

$$C_t^{o,l} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t^o - D_{t-1} \quad (\text{A.10})$$

$$C_t^{o,h} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t^o + B_{t-1}^{m,h} \frac{1 + i_{t-1}}{\Pi_t} \quad (\text{A.11})$$

where  $T_t^i$  are lump sum taxes per capita and  $Y_t$  is output per middle-generation household.<sup>64</sup>

Aggregate consumption in this economy is given by the following expression:

$$C_t = N_t C_t^y + N_{t-1} \left( \eta_s C_t^{m,l} + (1 - \eta_s) C_t^{m,h} \right) + N_{t-2} \left( \eta_s C_t^{o,l} + (1 - \eta_s) C_t^{o,h} \right)$$

## A.2 Firms' Problem, Labor Supply and Wage Determination

In this section, we specify the firm's problem in the baseline case with no capital accumulation. Firms choose labor to maximize profits subject to a standard decreasing returns to scale production function, taking wages as given:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t^d \quad (\text{A.12})$$

$$\text{s.t.} \quad Y_t = A_t \left( L_t^d \right)^\alpha \quad (\text{A.13})$$

where  $L_t^d$  is firm's labor demand. Firms' labor demand is determined by equating the real wage to the marginal product of labor:

$$\frac{W_t}{P_t} = \alpha A_t \left( L_t^d \right)^{\alpha-1} \quad (\text{A.14})$$

Each middle-generation household operates a firm and collects profits from its operation. The total measure of firms in the economy is  $N_{t-1}$ , and therefore grows with the total population. All firms are identical sharing the same labor share parameter  $\alpha$ .

Labor supply is exogenous and fixed over a household's lifetime. When population is constant ( $g = 0$ ), then labor supply is constant and can be normalized to unity. In the absence of downward nominal wage rigidity, the real wage equalizes labor supply to labor demand:

$$\left( N_t L^y + N_{t-1} \left( \eta_s L^{m,l} + (1 - \eta_s) L^{m,h} \right) + N_{t-2} L^o \right) = N_{t-1} L_t^{flex} \quad (\text{A.15})$$

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<sup>64</sup>Output is not expressed in per capita terms to avoid a proliferation of population growth rate terms. In this economy, aggregate output is  $N_{t-1} Y_t$  while the total population is  $N_t + N_{t-1} + N_{t-2}$ .

where  $w_t^{flex}$  defines the market-clearing real wage.

In the presence of downward nominal wage rigidity, the real wage may exceed the market clearing real wage. In this case, labor is rationed with a proportional reduction in labor employed across all households (i.e. if total labor demand is 10% below the full-employment level, then labor falls 10% for all cohorts).

We assume that nominal wages are downwardly rigid implying that real wages exceed the market-clearing level in the presence of deflation. The process determining the real wage is given below:

$$W_t = \max \left\{ \tilde{W}_t, P_t w_t^{flex} \right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1 - \gamma) P_t w_t^{flex} \quad (\text{A.16})$$

### A.3 Monetary and Fiscal Policy

Monetary and fiscal policy are straightforward. We assume a monetary policy rule of the following form:

$$1 + i_t = \max \left( 1, (1 + i^*) \left( \frac{\Pi_t}{\Pi^*} \right)^{\phi_\pi} \right) \quad (\text{A.17})$$

where  $i^*$  is the targeted natural rate and  $\Pi^*$  is the central bank's gross inflation target. If the central bank has the correct natural rate target  $i^*$ , then inflation is stabilized at  $\Pi = 1$  in steady state.

Taxation is determined by the government's budget constraint and exogenous processes for government spending, the public debt, and taxation of young households. We typically assume that the ratio of taxes between old and middle-generation households satisfies the following rule:

$$T_t^o = \beta \frac{1 + i_{t-1}}{\Pi_t} T_t^m \quad (\text{A.18})$$

In steady state, this fiscal rule ensures that changes in taxation have no effect on loan supply. We consider exceptions to this fiscal rule where taxes are levied only on old or middle-aged households respectively. The government's budget constraint together with the fiscal rule determines  $T_t^m$  and  $T_t^o$  in response to the other exogenous fiscal processes:

$$B_t^g + T_t^y (1 + g_t) + T_t^m + \frac{1}{1 + g_{t-1}} T_t^o = G_t + \frac{1}{1 + g_{t-1}} \frac{1 + i_{t-1}}{\Pi_t} B_{t-1}^g \quad (\text{A.19})$$

where all fiscal variables are all normalized in terms of per middle-generation quantities.

### A.4 Market Clearing and Equilibrium

Asset market clearing requires that total lending from savers equals total borrowing from credit constrained young households and poor middle-generation households. This condition is given

below:

$$(1 - \eta_s) N_{t-1} B_t^{m,h} = N_t \frac{D_t}{1 + r_t} + \eta_s N_{t-1} \frac{D_t}{1 + r_t} \quad (\text{A.20})$$

$$(1 - \eta_s) B_t^{m,h} = (1 + g_t + \eta_s) \frac{D_t}{1 + r_t} \quad (\text{A.21})$$

It can be verified that asset market clearing implies that aggregate consumption equals aggregate output less aggregate government purchases:

$$C_t = N_{t-1} Y_t - (N_t + N_{t-1} + N_{t-2}) G_t \quad (\text{A.22})$$

A competitive equilibrium is a set of aggregate allocations  $\{Y_t, C_t^{m,h}, C_t^{o,h}, B_t^{m,h}, L_t^{flex}, T_t^m, T_t^o\}_{t=0}^{\infty}$ , price processes  $\{i_t, \Pi_t, w_t, w_t^{flex}\}_{t=0}^{\infty}$ , exogenous processes  $\{G_t, g_t, D_t, T_t^y, B_t^g\}_{t=0}^{\infty}$  and initial values of household saving, nominal interest rate, real wage, and the public debt  $\{B_{-1}^{m,h}, i_{-1}, w_{-1}, B_{-1}\}$  that jointly satisfy:

1. Household Euler equation (A.6)
2. Household budget constraints (A.9) and (A.11)
3. Asset market clearing (A.21)
4. Fiscal policy rule (A.18)
5. Government budget constraint (A.19)
6. Monetary policy rule (A.17)
7. Full-employment labor supply (A.15)
8. Full-employment wage rate:  $w_t^{flex} = \alpha A_t \left(L_t^{flex}\right)^{\alpha-1}$
9. Labor demand condition:  $w_t = \alpha A_t \left(\frac{Y_t}{A_t}\right)^{\frac{\alpha-1}{\alpha}}$
10. Wage process:  $w_t = \max \left\{ \tilde{w}_t, w_t^{flex} \right\}$  where  $\tilde{w}_t = \gamma \frac{w_{t-1}}{\Pi_t} + (1 - \gamma) w_t^{flex}$

## B Inequality and the Natural Rate

There is no general result about how an increase in inequality affects the real rate of interest. This relationship will depend on how changes in income affect the relative supply and demand for loans. Here we show a simple example illustrating that there are relatively plausible conditions under which higher inequality will in fact reduce the natural rate of interest.



Consider first one form of income inequality, that of inequality across generations. As we have seen in (11), the relative endowment of the old versus the middle generation affects the real interest rate; moving resources from the old to the middle generation will increase savings and thereby put downward pressure on the real interest rate. The conclusion here is that redistribution that raises savings increases downward pressure on the real rate. Consider now the alternative: if all generations receive the same endowment  $Y_t^y = Y_t^m = Y_t^o$ , then it is easy to see that there is no incentive to borrow or lend, and, accordingly, the real interest rate is equal to the inverse of the discount factor  $1 + r_t = \beta^{-1}$ . It is thus inequality of income across generations that is responsible for our results and what triggers possibly negative real interest rates.

Generational inequality, however, is typically not what people have in mind when considering inequality; instead commentators often focus on unequal income of individuals within the working-age population. This type of inequality can also have a negative effect on the real interest rate. Before getting there, however, let us point out that this need not be true in all cases. Consider, for example, an endowment distribution  $Y_t(z)$ , where  $z$  denotes the type of an individual that is known once he is born. Suppose further that once again  $Y_t^y(z) = Y_t^m(z) = Y_t^o(z)$  for all  $z$ . Once again, income is perfectly smoothed across ages and the real interest rate is given by  $\beta^{-1}$ . The point is that we can choose any distribution of income  $Y(z)$  to support that equilibrium so that, in this case, income distribution is irrelevant for the type of preferences we have assumed.<sup>65</sup>

Let us now consider the case when inequality in a given cohort can in fact generate negative pressure on real interest rates. When authors attribute demand slumps to a rise in inequality they typically have in mind - in the language of old Keynesian models - that income gets redistributed from those with a high propensity for consumption to those that instead wish to save their income. We have already seen how this mechanism works in the case of inequality across generations. But we can also imagine that a similar mechanism applies if income gets redistributed within a cohort as long as some of the agents in that cohort are credit constrained.

Again let us assume that only middle aged and the old generation receive an income endowment. Now, however, suppose that some fraction of households receive a larger endowment in their middle generation (i.e. high-income households) while the remaining households (i.e. low-income households) receive a very small endowment in the middle period of life. For simplicity, all households receive the same income endowment in old age (this could be thought of as some sort of state-provided pension like Social Security). For sufficiently low levels of the middle-period endowment and a sufficiently tight credit constraint, low-income households will remain credit constrained in the middle period of life. These households will rollover their debt in the middle generation and only repay their debts in old age, consuming any remaining endowment. In this situation, only the high-income households will save in the middle period and will, therefore, supply savings to both credit-constrained middle-generation households and the youngest

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<sup>65</sup>More general preference specifications, however, can easily make a difference.

generation.

As before, we can derive an explicit expression for real interest rate in this richer setting with multiple types of households. Under the conditions described above, the only operative Euler equation is for the high-income households who supply loans in equilibrium. The demand for loans is obtained by adding together the demand from young households and from the credit constrained low-income households. The expression we obtain is a generalization of the case obtained in equation (11):

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t + \eta_s) D_t}{(1 - \eta_s) (Y_t^{m,h} - D_{t-1})} + \frac{1}{\beta} \frac{Y_{t+1}^o}{(Y_t^{m,h} - D_{t-1})} \quad (\text{A.23})$$

where  $\eta_s$  is the fraction of low-income households,  $Y_t^{m,h}$  is the income of the high-income middle-generation and  $Y_{t+1}^o$  is the income of these households in the next period (i.e. the pension income received by all households). If  $\eta_s = 0$  we recover the expression for the real interest rate derived in (11).

Total income for the middle generation is a weighted average of high and low income workers:  $Y_t^m = \eta_s Y_t^{m,l} + (1 - \eta_s) Y_t^{m,h}$ . Let us then define an increase in inequality as a redistribution of middle-generation income from low to high-income workers, without any change in  $Y_t^m$ . While this redistribution keeps total income for the middle generation constant by definition, it must necessarily lower the real interest rate by increasing the supply of savings which is only determined by the income of the wealthy. This can be seen in equation (A.23) where the real interest rate is decreasing in  $Y_t^{m,h}$  without any offsetting effect via  $Y_t^{m,l}$ .<sup>66</sup>

As this extension of our model suggests, the secular rise in wage inequality in recent decades in the US and other developed nations may have been one factor in exerting downward pressure on the real interest rate. Labor market polarization - the steady elimination of blue-collar occupations and the consequent downward pressure on wages for a large segment of the labor force - could show up as increase in income inequality among the working-age population, lowering the real interest rate in the manner described here (for evidence on labor market polarization, see e.g. Autor and Dorn (2013) and Goos, Manning and Salomons (2009)). Several other theories have been suggested for the rise in inequality. To the extent that they imply an increase in savings, they could fit into our story as well.

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<sup>66</sup>As emphasized earlier, not all forms of income inequality should be expected to have a negative effect on the real interest rate. If middle-generation income is drawn from a continuous distribution, a mean-preserving spread that raises the standard deviation of income could be expected to have effects on both the intensive and extensive margin. That is the average income among savers would rise, but this effect would be somewhat offset by an increase in the fraction of credit-constrained households (i.e an increase in  $\eta_s$ ). The extensive margin boosts the demand for loans and would tend to increase the real interest rate. Whether inequality raises or lowers rates would depend on the relative strength of these effects.

## C Linearization and Solution

In this section, we detail the linearization and general solution to the model without capital but with income received in all periods. For simplicity, we do not consider the effect of population growth shocks which greatly complicate the linearization and the computation of analytical solutions.

The generalized model with income received in all three periods and credit constrained middle-generation households can be summarized by the following linearized AD curve and linearized AS curve.

$$i_t = E_t \pi_{t+1} - s_y (y_t - g_t) + (1 - s_w) E_t (y_{t+1} - g_{t+1}) + s_w d_t + s_d d_{t-1} \quad (\text{A.24})$$

$$y_t = \gamma_w y_{t-1} + \gamma_w \frac{\alpha}{1 - \alpha} \pi_t \quad (\text{A.25})$$

where various coefficients are given in terms of their steady state values.

$$\begin{aligned} \gamma_w &= \frac{\gamma}{\bar{\pi}} \\ s_y &= \frac{\bar{Y}_{m,h}}{\bar{Y}_{m,h} - \bar{D}} \\ s_d &= \frac{\bar{D}}{\bar{Y}_{m,h} - \bar{D}} \\ s_w &= \frac{1 + \beta}{\beta} \frac{(1 + \bar{g} + \eta_s) \bar{D}}{\bar{i}/\bar{\pi} (\bar{Y}_{m,h} - \bar{D})} \end{aligned}$$

The exogenous shocks are the collateral shock  $d_t$  and the government spending shock  $g_t$ , which means that a solution to this linear system takes the form:

$$y_t = \beta_y y_{t-1} + \beta_g g_t + \beta_d d_t + \beta_{d,l} d_{t-1} \quad (\text{A.26})$$

$$\pi_t = \alpha_y y_{t-1} + \alpha_g g_t + \alpha_d d_t + \alpha_{d,l} d_{t-1} \quad (\text{A.27})$$

Solving by the method of undetermined coefficients, we obtain the following expressions for the coefficients that determine equilibrium output and inflation in response to collateral and gov-

ernment spending shocks.

$$\beta_y = 0 \quad (\text{A.28})$$

$$\alpha_y = -\frac{1-\alpha}{\alpha} \quad (\text{A.29})$$

$$\beta_{d,l} = \frac{s_d}{s_y + \frac{1-\alpha}{\alpha}} \quad (\text{A.30})$$

$$\alpha_{d,l} = \frac{1-\alpha}{\gamma_w \alpha} \beta_{d,l} \quad (\text{A.31})$$

$$\beta_d = \frac{s_w + \beta_{d,l} \left( \frac{1-\alpha}{\gamma_w \alpha} + (1-s_w) \right)}{s_y + (1-s_w) \rho_d + \frac{1-\alpha}{\alpha} (1 - 1/\gamma_w \rho_d)} \quad (\text{A.32})$$

$$\alpha_d = \frac{1-\alpha}{\gamma_w \alpha} \beta_d \quad (\text{A.33})$$

$$\beta_g = \frac{s_y + (1-s_w) \rho_g}{s_y + (1-s_w) \rho_g + \frac{1-\alpha}{\alpha} (1 - 1/\gamma_w \rho_g)} \quad (\text{A.34})$$

$$\alpha_g = \frac{1-\alpha}{\gamma_w \alpha} \beta_g \quad (\text{A.35})$$

By substituting (A.25) into (A.24), we can obtain a first-order difference equation in output. This forward-looking difference equation implies that inflation and output will be determinate if and only if the following condition obtains:

$$s_y - (1-s_w) > \frac{1-\alpha}{\alpha} \frac{1-\gamma_w}{\gamma_w}$$

When  $s_w = 1$ , this condition is the same determinacy condition as discussed in the main text. When the above condition holds, there is a unique rational expectations equilibrium in the deflation steady state. The left hand side is always positive, so in the case of perfect price rigidity (i.e.  $\gamma_w = 1$ ), this condition is satisfied and the deflation steady state is locally unique.

## D Properties of Secular Stagnation Equilibrium

Here we provide a formal proof for various properties of the secular stagnation equilibrium described in the body of the text.

**Proposition 1.** *If  $\gamma > 0$ ,  $\Pi^* = 1$ , and  $i^* = r^f < 0$ , then there exists a unique determinate secular stagnation equilibrium.*

*Proof.* Under the assumptions of the proposition, the inflation rate at which the zero lower bound binds given in equation (30) is strictly greater than unity. Let  $Y_{AD}$  denote the level of output implied by the aggregate demand relation and  $Y_{AS}$  denote the level of output implied by the

aggregate supply relation. For gross inflation rates less than unity,  $Y_{AD}$  and  $Y_{AS}$  are given by:

$$Y_{AD} = D + \psi\Pi \quad (\text{A.36})$$

$$Y_{AS} = \left( \frac{1 - \frac{\gamma}{\Pi}}{1 - \gamma} \right)^{\frac{\alpha}{1-\alpha}} Y^f \quad (\text{A.37})$$

where  $\psi = \frac{1+\beta}{\beta} (1+g) D > 0$ . The AD curve is upward sloping because  $\Pi < 1 < \Pi_{kink}$  under our assumptions and, therefore, the zero lower bound binds.

When  $\Pi = \gamma$ ,  $Y_{AD} > Y_{AS} = 0$ . When  $\Pi = 1$ , the real interest rate equals  $\Pi^{-1} = 1 > r^f$ . Thus, when  $\Pi = 1$ ,  $Y_{AD} < Y^f$ . Furthermore, from the equations above, when  $\Pi = 1$ ,  $Y_{AS} = Y^f$ . Therefore, it must be the case that  $Y_{AD} < Y_{AS}$  when  $\Pi = 1$ . Since the AS and AD curve are both continuous functions of inflation, it must be the case that there exists a  $\Pi_{ss}$  at which  $Y_{AD} = Y_{AS}$ .

To establish uniqueness, we first assume that there exist multiple distinct values of  $\Pi_{ss}$  at which  $Y_{AD} = Y_{AS}$ . In inflation-output space (output on the x-axis), the AS curve lies above the AD curve when inflation equals  $\gamma$  and the AS curve lies below the AD curve for inflation at unity - see equation (27). Thus, if multiple steady states exist, given that AS is a continuous function, there must exist at least three distinct points at which the AS and AD curve intersect.

At the first intersection point, the slope of AS curve crosses the AD line from above and, therefore, at the second intersection the AS curve crosses the AD curve from below. Since the AD curve is a line, the AS curve as a function of output is locally convex in this region. Similarly, between the second and third intersection, the AS curve is locally concave. Thus, given an increase in  $Y$ , the AS curve must first have a positive second derivative followed by a negative second derivative.

We compute the second derivative of inflation with respect output of the AS curve and derive the following expression:

$$\frac{d^2\Pi}{dY^2} = G(Y) \left( (1 + \phi) (1 - \gamma) \left( \frac{Y}{Y^f} \right)^\phi + (\phi - 1) \right) \quad (\text{A.38})$$

$$G(Y) = \frac{\phi\gamma(1 - \gamma) \left( \frac{Y}{Y^f} \right)^\phi}{Y^2 \left( 1 - (1 - \gamma) \left( \frac{Y}{Y^f} \right)^\phi \right)} \quad (\text{A.39})$$

$$\phi = \frac{1 - \alpha}{\alpha} \quad (\text{A.40})$$

As can be seen, over the region considered, the function  $G(Y)$  is positive and, therefore, the convexity of the AS curve is determined by the second term. This term may be negative if  $\phi < 1$ , but this expression is increasing in  $Y$  between 0 and  $Y_f$ . Therefore, the second derivative cannot switch signs from positive to negative. Thus, we have derived a contradiction by assuming multiple steady states. Therefore, there must exist a unique intersection point.

As established before, it must be the case that the AS curve has a lower slope than the AD

curve at the point of intersection. The slope of the AS curve is:

$$\frac{d\Pi}{dY} = \frac{1-\alpha}{\alpha} \frac{1}{\gamma} \frac{\Pi}{Y} (\Pi - \gamma) \quad (\text{A.41})$$

If the slope of the AS curve is less than the slope of the AD curve at the intersection point, then it must be the case that:

$$\begin{aligned} \frac{1-\alpha}{\alpha} \frac{\Pi}{Y} \left( \frac{\Pi}{\gamma} - 1 \right) &< \psi^{-1} \\ \frac{1-\alpha}{\alpha} \frac{\psi\Pi}{Y} \left( \frac{\Pi}{\gamma} - 1 \right) &< 1 \\ \frac{1-\alpha}{\alpha} \frac{Y-D}{Y} \left( \frac{\Pi}{\gamma} - 1 \right) &< 1 \\ s_y \frac{\alpha}{1-\alpha} + 1 &> \frac{\Pi}{\gamma} \\ \frac{\gamma}{\Pi} \left( s_y \frac{\alpha}{1-\alpha} + 1 \right) &> 1 \end{aligned}$$

The last inequality here is precisely the condition for determinacy discussed in Section 6. Thus, the unique secular stagnation steady state is always determinate as required.  $\square$

## E Calvo Pricing

In this section, we modify the aggregate supply block of our model to consider product market frictions instead of downward nominal wage rigidity. As in our baseline model, we assume that middle-generation households supply a constant level of labor  $\bar{L}$ . However, wages adjust frictionlessly to ensure that labor is fully employed in all periods.

Monopolistically competitive firms produce a differentiated good  $l$  and set nominal price periodically. Household consume a Dixit-Stiglitz aggregate of these differentiated goods implying that each firm faces the following demand schedule:

$$y_t(l) = Y_t \left( \frac{p_t(l)}{P_t} \right)^{-\theta} \quad (\text{A.42})$$

$$P_t = \left( \int p_t^{1-\theta} dl \right)^{\frac{1}{1-\theta}} \quad (\text{A.43})$$

where  $\theta$  is the elasticity of substitution in the Dixit-Stiglitz aggregator and  $P_t$  is the price level of the consumption bundle consumed by households. Production only depends on labor and labor market clearing requires total labor demand to equal labor supply:

$$y_t(l) = L_t(l) \quad (\text{A.44})$$

$$\bar{L} = \int L_t(l) dl \quad (\text{A.45})$$

Combining labor market clearing with the demand for each product (A.44), we can derive an expression for output in terms of exogenous labor supply and a term that reflects losses due to misallocation from pricing frictions:

$$Y_t = \frac{\bar{L}}{\Delta_t} \quad (\text{A.46})$$

$$\Delta_t = \int \left( \frac{p_t(l)}{P_t} \right)^{-\theta} dl \quad (\text{A.47})$$

Under Calvo pricing, firms are periodically able to reset their prices and will choose a single optimal reset price irrespective of the time since their last price change. Under the Calvo assumption, we can dynamic expressions for inflation and the misallocation term  $\Delta_t$  in terms of the reset price  $p_t^*$ :

$$1 = \chi \Pi_t^{\theta-1} + (1 - \chi) \left( \frac{p_t^*}{P_t} \right)^{1-\theta} \quad (\text{A.48})$$

$$\Delta_t = \chi \Pi_t^\theta \Delta_{t-1} + (1 - \chi) \left( \frac{p_t^*}{P_t} \right)^{-\theta} \quad (\text{A.49})$$

where  $\chi$  is the Calvo parameter - the fraction of firms that do not adjust prices in the current period. Equations (A.46), (A.48), and (A.49) collectively define the aggregate supply block of the model with monopolistic competition and price friction. The IS curve and monetary policy rule close the model.

In steady state, we can derive the long-run Phillips curve by coming the steady state cases of equations (A.46), (A.48), and (A.49). The steady state AS curve is given below:

$$Y = \bar{L} \frac{1 - \chi \Pi^\theta}{1 - \chi} \left( \frac{1 - \chi}{1 - \chi \Pi^{\theta-1}} \right)^{\frac{\theta}{\theta-1}}$$

## F Incorporating Money

In this section, we extend our baseline model to explicitly introduce a role for money and a money demand function. Households now have preferences for real money balances to capture the value of money in easing transactions frictions. For simplicity, we assume that households only hold money in the middle-period of life and utility over real money balances are separable. We also assume that there exists a level of real money balances  $\bar{m}$  at which households are satiated in money - that is  $v'(\bar{m}) = 0$ .

We specify and characterize the household's problem in the case of income received in the middle-period only and taxes paid in all periods. For household  $i$ , their objective function and

budget constraints are given below:

$$\max_{C_t(i), C_{t+1}(i), M_{t+1}(i), C_{t+2}(i)} \mathbb{E}_t \{ \log(C_t(i)) + \beta \log(C_{t+1}(i)) + \beta v(M_{t+1}(i)) + \beta^2 \log(C_{t+2}(i)) \} \quad (\text{A.50})$$

$$\text{s.t.} \quad C_t(i) = B_t(i) - T_t^y \quad (\text{A.51})$$

$$C_{t+1}(i) = Y_{t+1} - T_{t+1}^m + B_{t+1}(i) - M_{t+1}(i) - \frac{(1+i_t)}{\Pi_{t+1}} B_t(i) \quad (\text{A.52})$$

$$C_{t+2}(i) = \frac{1}{\Pi_{t+1}} M_{t+1}(i) - T_{t+2}^o - \frac{(1+i_{t+1})}{\Pi_{t+2}} B_{t+1}(i) \quad (\text{A.53})$$

$$B_{t+j}(i) \leq \mathbb{E}_{t+j}(1+r_{t+j+1}) D_{t+j} \quad \text{for } j = 0, 1 \quad (\text{A.54})$$

where  $M_{t+1}(i)$  are real money balances demanded by household  $i$ . Money earns zero interest and carries a liquidity premium on bonds away from the zero lower bound. The household's money demand condition is given below:

$$C_{t+1}(i) v'(M_{t+1}(i)) = \frac{i_{t+1}}{1+i_{t+1}} \quad (\text{A.55})$$

The above expression implicitly defines a money-demand equation. The given monetary policy rule determines real money balances via (A.55). Given a representative middle generation cohort and given that only the middle-generation demands money, we can drop the  $i$  and money demand per middle-generation household is:

$$M_t = v'^{-1} \left( \frac{i_t}{1+i_t} \frac{1}{C_t^m} \right) \quad (\text{A.56})$$

The issuance of money by the central bank modifies the government's budget constraint in (A.19). The government's consolidated budget constraint expressed in real terms is given below:

$$B_t^g + M_t + T_t^y(1+g_t) + T_t^m + \frac{1}{1+g_{t-1}} T_t^o = G_t + \frac{1}{1+g_{t-1}} \left( \frac{1+i_{t-1}}{\Pi_t} B_{t-1}^g + \frac{1}{\Pi_t} M_{t-1} \right) \quad (\text{A.57})$$

We assume a fiscal policy that adjust taxes  $T_t^y, T_t^m$ , and  $T_t^o$  to keep the government's consolidated liabilities,  $M_t + B_t^g$  at some constant target level. In particular, this means that, in periods of deflation, the nominal stock of government liabilities is being reduced in proportion to the fall in the price level. In the steady state of a stagnation equilibrium featuring a constant rate of deflation, nominal government liabilities are contracting at the rate of deflation. Under a fiscal policy that keeps real government liabilities constant, the presence of money does not materially alter our conclusions.

## G Productivity Growth and Hysteresis

In this section, we extend the baseline model to include trend productivity growth and offer a simple extension to model hysteresis - where output gaps feedback onto the productivity growth



process. The extension of the model to include productivity growth does not greatly alter the basic features of the model but will allow the model to better match the dynamics of real GDP per capita. The aggregate demand block is still summarized by an asset market clearing that relates middle-generation income and the real interest rate:

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t) D_t}{Y_t^m - D_{t-1}} \quad (\text{A.58})$$

$$= \frac{1 + \beta}{\beta} \frac{(1 + g_t) \tilde{D}_t}{\tilde{Y}_t^m - \tilde{D}_{t-1} \frac{A_{t-1}}{A_t}} \quad (\text{A.59})$$

where  $\tilde{X}_t = X_t/A_t$  are detrended variables. So long as the collateral constraint grows at the same rate as productivity growth, there exists a balanced growth path with a constant real interest rate in steady state and quantities growing at the rate of productivity growth. Relative to the AD curve in the baseline model, the only difference is that higher productivity growth increases saving by lowering the value of debt incurred when young.

Trend productivity growth also impacts the wage norm as the flexible price real wage rises over time. Now, deflation must exceed the rate of productivity growth for the wage norm to bind. More generally, if nominal wages are indexed to the inflation target, the shortfall of inflation below target must exceed the growth rate of productivity. The wage norm indexed to inflation is given below along with the flexible-price real wage:

$$W_t = \max \left\{ \gamma \Pi^* W_{t-1} + (1 - \gamma) P_t w_t^{flex}, P_t w_t^{flex} \right\} \quad (\text{A.60})$$

$$w_t^{flex} = \alpha A_t \bar{L}^{\alpha-1} \quad (\text{A.61})$$

Real wages and output can be detrended by productivity growth to obtain stationary variables. Trend stationary real wages are given by the following expression:

$$\tilde{w}_t = \max \left\{ \gamma \Pi^* \frac{\tilde{w}_{t-1}}{\Pi_t} \frac{A_{t-1}}{A_t} + (1 - \gamma) \tilde{w}^{flex}, \tilde{w}^{flex} \right\} \quad (\text{A.62})$$

where  $\tilde{w}_t$  are detrended real wages and  $\tilde{w}^{flex} = \alpha \bar{L}^{\alpha-1}$ . In steady state, the wage norm binds when  $\Pi^* > \bar{\Pi} \mu$  where  $\mu$  is the steady state growth rate of productivity. The AS curve can be derived by substituting the following expressions for output and the full-employment level of output:

$$\tilde{w}_t = \alpha \tilde{Y}_t^{\frac{\alpha-1}{\alpha}} \quad (\text{A.63})$$

$$\tilde{w}^{flex} = \alpha \tilde{Y}_{fe}^{\frac{\alpha-1}{\alpha}} \quad (\text{A.64})$$

Equations (A.59), (A.62), and (A.63) along with the Fisher relation and the monetary policy rule jointly determine  $\{\tilde{Y}_t, \tilde{w}_t, r_t, \Pi_t, i_t\}$ .<sup>67</sup>

<sup>67</sup>Income for the middle period household  $\tilde{Y}_t^m$  is assumed to be a constant fraction of total income.

The modified equilibrium conditions presented in this section have simply taken productivity growth as an exogenous process. One possibility is that prolonged output gaps feedback into slower productivity growth. Productivity growth could be related to the output gap simply by positing a simple feedback process. This feedback process represents a reduced form mechanism whereby prolonged output gaps reduced productivity growth by, for example, reducing investment, technology adoption, expenditures in public or private research and development, or limiting the degree of firm entry. We posit the following feedback rule:

$$\frac{A_t}{A_{t-1}} = \mu_0 \left( \frac{\tilde{Y}_t}{\tilde{Y}_{fe}} \right)^\kappa$$

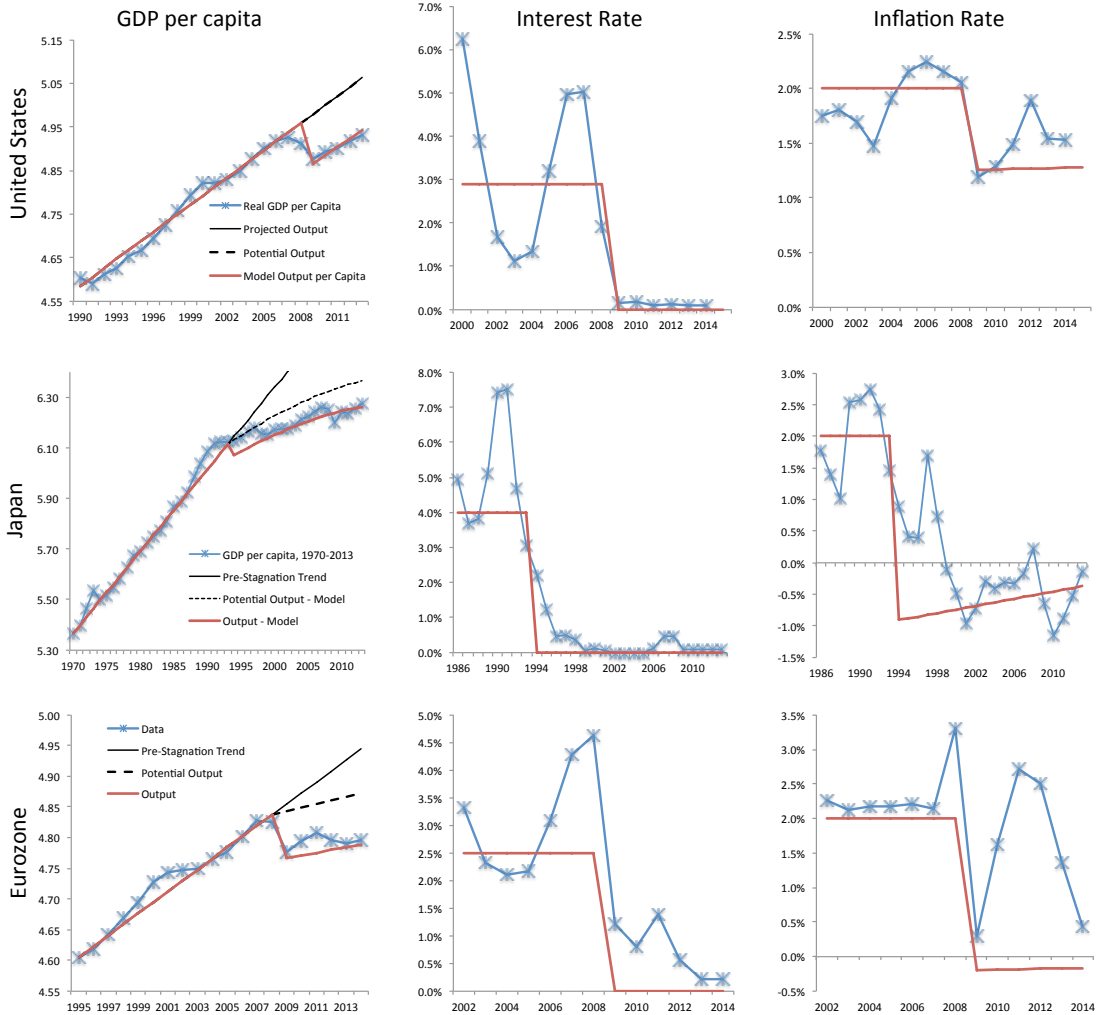
where  $\kappa$  determines the strength of the hysteresis effect.

## H Quantitative Calibration: US, Europe and Japan

The simple three period OLG model captures the salient features of secular stagnation: persistently low levels of inflation and interest rates and below trend output. Though our model is obviously highly stylized, we still think it is of value to explicitly parameterize it and examine its capacity to explain recent stagnation episodes. Figure A.1 display the key series whose behavior our theory is trying to explain. In all these episodes, we have witnessed a drop in the short-term nominal interest rate to close to zero and a decline in inflation below the implicit inflation target of the central bank. The size of the fall in inflation has varied. Japan has experience outright deflation, but in US and Europe, deflation was very short-lived while inflation has remained persistently below the target of the central banks. The size of the output gap is more controversial. Here, we calibrate the model to recent estimates, but, conceptually, even without any remaining output gap, low natural rates of interest pose an ongoing challenge for monetary policy.

To evaluate the capacity of our model to match the behavior of GDP per capita, interest rates, and inflation, we choose parameters to target steady state output gaps and deviations of inflation from target for each region. As discussed in the previous section (Appendix G), we modify the equilibrium conditions of our model to allow for trend productivity growth and wage indexation to the inflation target. In the case of Japan and the Eurozone, we assume that trend productivity growth fell at the onset of secular stagnation to fit the slower trend rate of GDP growth in each region. We log-linearize the model around the secular stagnation steady state, and plot transition paths for output per capita, nominal interest rates, and inflation in response to a shock to the collateral constraint in each region. Given the three period OLG structure, each period is taken to be 20 years.

Figure A.1: Data v. model transition paths: US, Japan, Eurozone



## H.1 US Calibration

Table 1 shows parameters annualized for the US (hence in computing steady state, these values need to be converted to 20 years). We pick an inflation target of 2 percent per year consistent with the Federal Reserve's implicit inflation target. Moreover, we assume that the wage norm is indexed at the inflation rate times the growth rate of productivity. In this case, the wage norm binds whenever inflation falls below target. We assume trend productivity growth of 2.1% per year to match average GDP per capita growth from 1990-2007. The rate of time preference  $\beta$  is set at a conventional value of 0.96. We set US population growth of 0.7% per year based upon UN population projections. We set  $\alpha$  at 0.7 to match the labor share.

The two novel parameters to choose are the degree of wage rigidity determined by  $\gamma$  and the collateral constraint on the young  $D$ . We must also set an initial value for the collateral constraint

**Table A.1:** Parameter values for US

Description	Parameter	Value
Population growth	$g$	0.7%
Discount rate	$\beta$	0.96
Labor share	$\alpha$	0.7
Inflation target	$\Pi^*$	2.0%
Pre-shock collateral (% of annual GDP)	$\frac{1+g}{1+r_{ini}} \frac{D_{ini}}{Y}$	126%
Post-shock collateral (% of annual GDP)	$\frac{1+g}{1+r_{ss}} \frac{D_{ss}}{Y}$	100%
Wage adjustment	$\gamma$	0.94
Youth income share	$Y_y/Y$	0.4

as the initial point for the determining the transition path of the economy. To find realistic values of the collateral constraint, it is necessary to include other sources of demand. We do this by assuming that some income is received by the young, the government issues public debt, and the government absorbs some output through purchases that are financed by taxes levied on the middle aged. Government spending as a percentage of GDP is set at 20% and public debt as a percentage of annual GDP is set at 100%. The government budget constraint determines the level of taxes.

Given government spending, taxes, and public debt, the collateral constraint  $D$ , the income distribution between young and old, and wage rigidity parameter are set to match the following targets: an output gap of 13%, an inflation rate of 1.4%, and household debt of 100% of GDP in 2014. The output gap represents the deviation of output per capita from its 1990-2007 trendline. The inflation rate is based on the growth rate of core PCE in 2014. The household debt target is taken from the Federal Reserve Flow of Funds and is set at 100% of GDP to match the sum of loans to households and nonprofits (13.7 trillion in 2014) and loans to nonfinancial noncorporate businesses (4.4 trillion in 2014). Finally, the pre-shock level of the collateral constraint is set to match the average nominal interest of 2.9% between 2001-2007.

How reasonable are the values we set for these parameters? The implied weight on last period wages ( $\gamma$  multiplied by the inflation target and productivity growth) is 0.98 per year. **Schmitt-Grohé and Uribe (2016)** provide an authoritative overview of the evidence on downward nominal wage rigidity and estimate a very similar wage adjustment curve as we use here. The implied income distribution implies 60% of national income is received by the middle-aged while 40% of national income is received by the young. Our calibration implies an initial level of household debt of 126% of GDP which is quite close to the pre-2008 level of loans to households and small businesses of 121% of GDP pre-2008. Therefore, the collateral shock we choose is quite close in magnitude to the contraction of lending experienced during the Great Recession.

**Table A.2:** Parameter values for Japan and Eurozone

Description	Parameter	Japan	Eurozone
Population growth	$g$	0%	0%
Discount rate	$\beta$	0.96	0.96
Labor share	$\alpha$	0.7	0.7
Inflation target	$\Pi^*$	2.0%	2.0%
Pre-shock collateral	$D_0$	0.37	0.27
Post-shock collateral	$D_1$	0.23	0.21
Wage adjustment	$\gamma$	0.90	0.89
Hysteresis elasticity	$\kappa$	4.8	3.0

The top row of Figure A.1 plots the transition paths for GDP per capita, inflation, and the interest rate over the relevant data series. A collateral shock can indeed explain the drop in inflation and the interest rate in our model. The nominal interest rate drops to the zero lower bound on impact while the inflation rate slightly overshoots gradually rising towards its steady state value of 1.4%. Output per capita falls by less on impact than in the long-run. Therefore, the output gap is widening slightly along the transition path. Indeed, this is consistent with the behavior of US GDP per capita which is drifting slightly away from the pre-recession trendline. Broadly speaking, our fairly simple model with a shock that matches the magnitude of the contraction in lending can capture the behavior of GDP per capita, inflation, and interest rates in the US over the Great Recession.

## H.2 Eurozone and Japan Calibration

With hysteresis effects, we can calibrate the model to analyze stagnation episodes in the Eurozone and Japan. As with the US calibration, we set the rate of time preference  $\beta$  and the labor share  $\alpha$  to standard values. Population growth in both regions is set to zero to reflect recent population trends. The inflation target is set at 2% in both regions. The key remaining parameters are the collateral constraint  $D$ , the degree of wage rigidity  $\gamma$ , and the hysteresis parameter  $\kappa$ . For both regions, we set these parameters to match the output gap, inflation rate, and change in trend output growth. The pre-shock level of the collateral constraint is set to match the nominal interest rate prior to the stagnation episode.<sup>68</sup> In each case, output is normalized to unity.

In the case of Japan, we target an output gap of 10%, a rate of deflation of -0.25%, and a reduction in trend productivity growth from 3.3% to 0.7%. The last value is determined by trend GDP per capita growth from 1970-1994 and from 1994-2008 respectively. Data on real GDP per

<sup>68</sup>For this calibration, we set government debt/spending to zero, assume income is received only in the middle period, and do not seek to match the collateral constraint to measures of household debt in the Eurozone or Japan.

capita come from the World Economic Indicators. The output gap is based on estimates from [Hausman and Wieland \(2014\)](#), and inflation rate is the growth rate of Japan's CPI as reported in the OECD main economic indicators. As Table A.2 shows, the implied degree of wage rigidity is low (relevant parameters are annualized). Because of hysteresis effects in our model, real wages are relatively flexible in comparison to the values found in [Schmitt-Grohé and Uribe \(2011\)](#). Hysteresis effects limit downward pressure on prices due to the paradox of toil, and therefore, limit the degree of deflation despite a persistent output gap.

In the case of the Eurozone, we target an output gap of 10%, an inflation rate of 0%, and a slowdown in productivity growth from 1.8% to 0.2%. The output gap is chosen to best fit the observed decline in output per capita during the Great Recession. The inflation rate reflects the growth rate of the harmonized CPI in the Eurozone in 2014. The slowdown in productivity growth is estimated by simple trendlines for GDP per capita growth from 1995-2008 and 2009-2014 respectively. The degree of wage rigidity in the Eurozone is comparable to values for Japan, and the hysteresis effects are about half as strong reflecting the more modest slowdown in trend GDP growth before and after the stagnation episode. Like the US, the output gap in the Eurozone is gradually increasing over time along the transition path consistent with the behavior of EU GDP per capita post Great Recession.

## I Derivation of the Quantitative Lifecycle Model

### I.1 Demographics and Labor Supply

The economy consists of a large number of households with identical utility parameters. Households enter economic maturity at age 26 after which they work, consume, have children, and participate in markets. Finally they die at age  $J$ , which we take to be 81 years. Households have children at age 26, and the population growth rate is determined by the total fertility rate ( $\Gamma$ ) of every family. People have a probability of dying stochastically before reaching maximum age  $J$ . The probability of surviving between age  $j$  and  $j + 1$  is denoted by  $s_j$ .<sup>69</sup> The unconditional probability of reaching age  $j$  is denoted with a superscript  $s^j$ .<sup>70</sup>

The total population alive at any given time  $N_t$  is the sum of the population of the individual ages  $N_{j,t}$ . The population size of a given generation  $N_{j,t}$  the population of the generation the previous year that has survived. But what about at model age 26 (the first year of economic maturity)? The total population of a generation entering economic majority at time  $t$  is equal to the total population of their parents when they entered economic maturity at time  $t - 25$ , times the total fertility rate of their parents generation. Thus population evolves in the model according

<sup>69</sup>Age specific survival rates may also vary over time  $t$ , however for notational simplicity we omit these additional subscripts.

<sup>70</sup>This can be calculated as the production on one-period survival probabilities:  $s^j = \prod_{m=26}^{j-1} s_m$ .

to the law of motion given below:

$$\begin{aligned}
N_t &= \sum_{j=26}^J N_{j,t} \\
N_{j+1,t+1} &= s_{j,t} N_{j,t} \text{ for } j \in \{27, J\} \\
N_{26,t} &= N_{26,t-25} * \Gamma_{26,t-25}
\end{aligned}$$

The total fertility rate along with the age at which households have children determines the rate of population growth rate. In a steady state, for a given total fertility rate  $\Gamma$ , the rate of population growth  $n$  is equal to:

$$n = \Gamma^{\frac{1}{25}} - 1 \quad (\text{A.65})$$

In a steady state with rate of population growth  $n$ , each generation is  $(1 + n)$  times larger than the previous. Thus the total population size is can be calculated as (normalizing the total population to 1):

$$N = \sum_{j=26}^J N_j \quad (\text{A.66})$$

$$N_{j+1} = s_j \frac{N_j}{(1+n)} \text{ for } j \in \{27, J\} \quad (\text{A.67})$$

$$N_{26} = \bar{N}_{26} \quad (\text{A.68})$$

Where  $\bar{N}_{26}$  is the normalized population in period 1, equal to

$$\bar{N}_1 = \frac{1}{\sum_{j=26}^J \frac{s_j}{(1+n)^{j-1}}}$$

Each household has an identical schedule of lifetime exogenous labor productivity, or human capital, denoted by  $hc_j$  which varies by age. Household receive no wage income after retirement, which in our model occurs after age 65 (model age 40). We assume labor is supplied inelastically. Therefore, wage income at full employment is equal to the wage multiplied by the individual age specific labor productivity  $hc_j$  net of labor taxes  $(1 - \tau^w)$ . In a secular stagnation, labor demand falls below labor supply, and labor is rationed proportionally for each cohort.

$$L_t^s = \sum_{j=26}^J N_{j,t} hc_j \quad (\text{A.69})$$

## I.2 Household's Problem

Households receive utility from two sources: (1) consumption, which is given by a time-separable constant elasticity of substitution (CES) utility function  $u(\cdot)$  with an elasticity of intertemporal substitution parameter  $\rho$ , and (2) bequests, which are divided equally among all descendants. The

bequest motive is also characterized by a CES function  $v(\cdot)$  whose argument is the amount of bequests left *per descendent*<sup>71</sup>, denoted by  $x$ . The utility of bequests is multiplied by a parameter  $\mu \geq 0$  which determines the strength of the bequest motive. Denoting consumption of household of age  $j$  at time  $t$  by  $C_{j,t}$  and the discount rate by  $\beta$ , a household who is born at time  $t$  then maximizes their lifetime expected utility:

$$U_t = \sum_{j=26}^J s^j \beta^{j-1} u(C_{j,t+j-1}) + s^J \beta^{J-1} \mu v(x_{J,t+J-1}) \quad (\text{A.70})$$

The household of age  $j$  can purchase or borrow a real asset  $a_{j,t}$  at price  $\xi_t$  at time  $t$  which is used as productive capital. At time  $t + 1$  it will pay the return  $r_{t+1}^k$  which is the rental rate of capital, and has a resell value(net of depreciation)  $(1 - \delta)\xi_{t+1}$ , where  $\xi_{t+1}$  is the relative price of capital in terms of the consumption good. All households, prior to the terminal period, participate in a perfectly competitive one period annuity market as in [Ríos-Rull \(1996\)](#). For a given cohort, the assets of households that die are distributed evenly to the surviving members of the cohort. Households also receive income from the pure profits from firms, denoted by  $\Pi_{j,t}$ .<sup>72</sup> Finally, the household may receive a bequest  $q_{j,t}$ . Bequest received are assumed to zero at all times except at age  $J - 24$ . Bequest given are zero at all time except at age  $J$ .

The flow budget constraint of a household of age  $j$  at time  $t$  can be then be written as:

$$c_{j,t} + \xi_t a_{j+1,t+1} + \Gamma_{26,t-j+26} \cdot x_{j,t} = (1 - \tau^w) w_t h c_j + \Pi_{j,t} + [r_t^k + \xi_t (1 - \delta)] \left( a_{j,t} + q_{j,t} + \frac{1 - s_j}{s_j} a_{j,t} \right) \quad (\text{A.71})$$

Household may wish to borrow against future income, and face a borrowing constraint of the same form as in the three-period model<sup>73</sup>

$$a_{j,t} \geq \frac{D_t}{1 + r_t} \quad (\text{A.72})$$

There is one further complication with bequests. Since there is stochastic mortality, not all parents will survive to their maximum life age in order to give bequests to their children. Thus, absent any insurance markets, there would be stochastic within-generation inequality, as some households would receive bequests and others would not. In order to remove this channel, we assume all generations participate in a form of bequest insurance markets. At the maximum age, all surviving members of a generation pool their optimal bequests, and divide them equally among their surviving children. Thus the relationship between bequests given (at age  $J$ ) and those received by children (at model age  $k$ ) at time  $t$  is given by:

$$q_{k,t} = \frac{N_{J,t-1} x_{J,t-1} \cdot \Gamma_{26,t-J+26}}{N_{k,t}} \quad (\text{A.73})$$

<sup>71</sup>Thus the total size of the bequest left by households is the bequest  $x$  multiplied by the fertility of the household.

<sup>72</sup>We assume that these are distributed proportional to labor income.

<sup>73</sup>We assume that  $D_t$  grows at the rate of productivity growth and is expressed in terms of consumption goods to ensure balanced growth.



### I.3 Final Goods Firms

There exist a continuum of final goods firms of type  $i$  of measure one that costlessly differentiate an intermediate good and resell to the representative household. The final good composite is the CES aggregate of these differentiated final goods:

$$Y_t = \left[ \int_0^1 y_t^f(i)^{\frac{\theta_t-1}{\theta_t}} di \right]^{\frac{\theta_t}{\theta_t-1}}$$

These firms are monopolistically competitive, set prices each period, and face a demand curve that takes the following form:

$$y_t^f(i) = Y_t \left( \frac{p_t(i)}{P_t} \right)^{-\theta_t}$$

where  $\theta_t$  is a time-varying shock to the firm's market power. An increase in  $\theta_t$  decreases a firm's market power and lowers equilibrium markups. Each final good producer uses  $y_t^m$  of intermediate good to produce output, according to a linear technology function  $y_t^f = y_t^m$ . A final good firm chooses real prices  $\frac{p_t(i)}{P_t}$  and  $y_t^f(i)$  to maximize real profits, subject to the production constraint:

$$\begin{aligned} & \max \frac{p_t(i)}{P_t} y_t^f(i) - \frac{p_t^{int}}{P_t} y_t^f(i) \\ & \text{subject to } y_t^f(i) = Y_t \left( \frac{p_t(i)}{P_t} \right)^{-\theta_t} \end{aligned}$$

where  $\frac{p_t^{int}}{P_t}$  is the price of the intermediate good taken as given by the firm.

The optimality condition for the real price of the firm's good is a time-varying markup over the price of the intermediate good:

$$\frac{p_t(i)}{P_t} = \frac{\theta_t}{\theta_t - 1} \frac{p_t^{int}}{P_t} \quad (\text{A.74})$$

The nominal price index is given by the following expression: and implies the following expression for the price of intermediate goods:

$$P_t = \left( \int p_t(i)^{1-\theta_t} di \right)^{\frac{1}{1-\theta_t}}$$

Since the price of the intermediate good is the same, all final goods firms make the same pricing decisions (no pricing frictions), and thus  $p_t(i) = P_t$ , thus we have:

$$\frac{p_t^{int}}{P_t} = \frac{\theta_t - 1}{\theta_t} \quad (\text{A.75})$$

With a retail elasticity of substitution of  $\theta_t$ , aggregate profits in equilibrium will be given by

$$\Pi_t = \frac{Y}{\theta} \quad (\text{A.76})$$

Profits from monopolistically competitive firms are distributed according to wage income. In equilibrium the total distributed profit must equal total profits:

$$\frac{Y_t}{\theta_t} = \sum_{j=26}^J N_{j,t} \Pi_{j,t} \quad (\text{A.77})$$

#### I.4 Intermediate Goods Firms

There exists a perfectly competitive intermediate goods sector that sells their production to the final goods sector at real price  $\frac{p_t^{int}}{P_t}$ . These firms operate a CES production function with an elasticity of substitution  $\sigma$ , hire labor, and rent capital. The representative intermediate good firms maximizes static real profits given the following production function:

$$\Pi_t^{int} = \max \frac{p_t^{int}}{P_t} Y_t - w_t L_t - r_t^k K_t \quad (\text{A.78})$$

$$Y_t = \left( \alpha (A_{k,t} K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) (A_{l,t} L_t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad (\text{A.79})$$

Labor productivity  $A_{L,t}$  grows at each period at the rate of  $g_t$ .

The first order conditions that determine labor and capital demand are given below:

$$w_t = \frac{p_t^{int}}{P_t} (1-\alpha) A_{l,t} \left( \frac{Y_t}{L_t} \right)^{\frac{1}{\sigma}} \quad (\text{A.80})$$

$$r_t^k = \frac{p_t^{int}}{P_t} \alpha A_{k,t} \left( \frac{Y_t}{K_t} \right)^{\frac{1}{\sigma}} \quad (\text{A.81})$$

The risk-free real rate is related to the return on capital by a standard no arbitrage condition:

$$1 + r_t = \frac{r_t^k + (1-\delta) \xi_t}{\xi_{t-1}} \quad (\text{A.82})$$

#### I.5 Relative Price of Capital Goods

Investment specific productivity is defined as the amount of capital goods that can be produced with one unit of investment. In particular, we will assume that capital goods are produced by perfectly competitive firms in an investment-specific production sector, who convert the final composite good into capital goods. These firms maximize the following profit function subject to a linear production function:

$$\Pi^K = \xi_t * K_t - Y_t^K \quad (\text{A.83})$$

$$K_t = z_t Y_t^K \quad (\text{A.84})$$

where  $\xi_t$  is the relative price of capital goods. Here,  $z_t$  is the productivity of the capital producing sector. The zero profit conditions mean that in equilibrium  $\xi_t = \frac{1}{z_t}$ . Thus the more productive is the investment goods sector, the lower the relative price of capital goods.

The aggregate capital stock evolves according to the standard law of motion:

$$K_{t+1} = (1 - \delta)K_t + \frac{I_t}{\xi_t} \quad (\text{A.85})$$

where  $\delta$  is the rate of depreciation,  $I_t$  is investment, and  $\xi_t$  is the relative price of capital goods.

The government spends an exogenous  $G_t$  and may accumulate debt. The budget constraint is given by:

$$b_{g,t} = G_t + (1 + r_t)b_{g,t-1} - T_t \quad (\text{A.86})$$

where the total tax bill is collected with labor income tax. For the purpose of our simulations, fiscal policy will be specified as an exogenous sequence of two variables: government debt to GDP and government spending to GDP. The wage income tax will then be endogenously determined by the model in order for the budget constraint to hold.

We have economized on notation by omitting real and nominal bonds as assets, they enter in the same way as in the simpler model so that there is both a well defined real interest rate  $r_t$  on a risk-free one period bond and a nominal interest rate  $i_t$ .

## I.6 Equilibrium

A competitive equilibrium is a set of household allocations:  $\{\{c_{j,t}, a_{j,t}, \Pi_{j,t}, q_{j,t}, x_{j,t}\}_{j=26}^J\}_{t=0}^\infty$ , a set of aggregate quantities  $\{Y_t, K_t, L_t, \Pi_t\}_{t=0}^\infty$ , a set of prices  $\{w_t, r_t^k, r_t, \frac{p_t^{int}}{P_t}, \xi_t\}_{t=0}^\infty$ , a set of government variables  $\{b_t, \tau_t^w, \tau_t^r\}_{t=0}^\infty$ , and a set of exogenous processes  $\{\{N_{j,t}, p_{j,t+1}\}_{j=0}^J\}_{t=0}^\infty$  that jointly satisfy:

1. Consumption and bequests maximizes (A.70) subject to (A.71) and (A.72)
2. Asset holdings satisfy (A.71), with  $a_1 = 0$  given.
3. Profits are distributed proportionally to labor income
4. Bequests received equal bequests given by the surviving parents according to (A.73)
5. Output is given by aggregate production function (A.78)
6. Population by age group is given by (A.66)
7. Aggregate labor supply is given by (A.69)
8. Aggregate profits are given by (A.76)
9. There are perfect factor markets, and thus (A.81), (A.80), and (A.82) hold.
10. Markup condition: (A.74)
11. The government satisfies budget equation (A.86)

12. Asset markets clear, and thus

$$\sum_{j=1}^J N_{j,t} \xi_t a_{j,t} = \xi_t K_t + b_t \quad \forall t$$

## I.7 Stationary Equilibrium

A steady state equilibrium is the same as the above definition, except that all variables are constant rather than subscripted by time. With productivity and population growth greater than zero, however, the economy is not stationary. However, since preferences are of the CRRA variety, the economy can be rewritten as stationary by applying a transformation. All cohort level variables are divided by  $(1+g)^t$ , aggregate variables are divided by  $(1+g)^t(1+n)^t$ , and wages are divided by  $(1+g)^t$ .

In a steady state equilibrium, there are  $4 * J + 12$  unknowns:  $J$  consumption demands  $c_j$ ,  $J$  asset demands  $a_j$ ,  $J$  shares of aggregate profits  $\Pi_j$ ,  $J$  age specific population sizes  $N_j$ , 1 bequest given  $x$ , 1 bequest received  $q$ , aggregate output  $Y$ , aggregate capital  $K$ , aggregate labor  $L$ , aggregate profits  $\Pi$ , wage rate  $w$ , return on capital  $r^k$ , interest rate  $r$ , price of intermediate goods  $\frac{p^{int}}{P}$ , relative price of capital  $\xi$ , and tax rate  $\tau^w$ .

There are also  $4 * J + 12$  equations:

1. (J-1) Euler equations
2. (1) Optimal bequest given equation
3. (1) Equation determining size of bequest received
4. (J) Flow budget equations
5. (J) Profit share equations
6. (1) Initial assets are zero
7. (J) Population equations
8. (1) Aggregate production function
9. (1) Asset market clearing condition
10. (1) Aggregate labor supply equation
11. (1) Aggregate profit Equation
12. (1) Wage equation
13. (1) Price of intermediate goods equation

14. (1) Rental rate of capital equation
15. (1) Relative price of capital is exogenously given
16. (1) Interest rate equation
17. (1) Government budget equation

## J Computational Method

Solving the quantitative model is implemented much along the lines as described in chapter 4 of [Auerbach and Kotlikoff \(1987\)](#). We begin by describing how to solve for variables in the stationary equilibrium. In solving for the stationary equilibrium, the algorithm begins with a guess for a subset of endogenous variables. For the purposes of the rest of the iteration, this subset of variables are treated as exogenous. This simplification makes the resulting system easier to solve for the endogenous variables, including the variables for which guesses were made.

We begin with a guess for the aggregate capital stock, the wage tax, and bequests received. Given the aggregate capital and labor stock, we can calculate interest rates and wages, as well as aggregate output and profits. This allows us to calculate optimal household behavior for consumption, asset holdings, and bequests given. Since the optimization problem of the household includes a debt limit, we cannot solve the household's optimal decisions analytically, as in [Auerbach and Kotlikoff \(1987\)](#). Instead, we use Matlab's convex optimization solver 'fmincon'.

In the steady state, the population of each age group is given by equation ([A.66](#)). Using the population sizes of each age group, we aggregate the optimal asset supply to obtain a new value of the aggregate capital stock. Using aggregate capital and labor, we calculate aggregate output and profits in the economy. We also update the tax rate to satisfy the government's budget constraint. Finally, we set bequests received to be equal to total bequests given by the individuals that survive to eighty-one divided by the number of children who survived to the bequest age of fifty-seven. Typically, 20-30 iterations are necessary to converge to the stationary equilibrium.

The approach used to solve for the transition path is similar to solving for the initial and final steady states of the model. We assume that the economy will be in a steady state after 150 years in the transition. In the years between the initial and final steady state, we must solve for all endogenous variables, including the optimal consumption and bequests choices for individuals of all generations. We assume the year 1 is in steady state, and that in the year 2, agents are surprised by an unexpected change in the *path* of key economic variables, such as productivity growth, total fertility rate, mortality profiles, etc. Their asset choices were made previously in period 1, and they must now adjust their consumption and saving choices to the path of these variables. At the time of the shock, households have perfect foresight.

Our algorithm now begins with a guess for the same subset of endogenous variables, but now the guess for each variable is a  $150 \times 1$  vector for each year of the transition path. Given the vector of guesses for the capital stock, we can calculate a vector of prices and wages, and thus optimally solve for consumption and bequests of each generation in the transition period. After the year 151, agents assume the economy will be in the final steady state. We then aggregate the individual asset supply decisions to get a new value for capital supply, and repeat the algorithm until convergence.

## K Extension with Financial Frictions

All borrowing and lending is done through a perfectly competitive banking system. The banking system takes lending from individuals and matches them with borrowers, who can be either firms or other individuals. For every 1 unit of either investment or final good which is lent, it costs the bank  $\phi$ . Because there is perfect competition in the banking industry, the rate the bank charges to borrowers,  $r_b$ , and the rate the bank gives to lenders,  $r_l$  is related by the following arbitrage equation:

$$r_b = r_l + \phi \quad (\text{A.87})$$

### K.1 Individuals

Individuals have the same utility function, budget constraint, and borrowing limit as they previously did. However, now the interest rate that they face depends on whether they are a borrower or a lender: if  $a_t < 0$ , they face the borrowing interest rate  $r_b$ . If  $a_t > 0$ , they face the lending interest rate  $a_l$ .

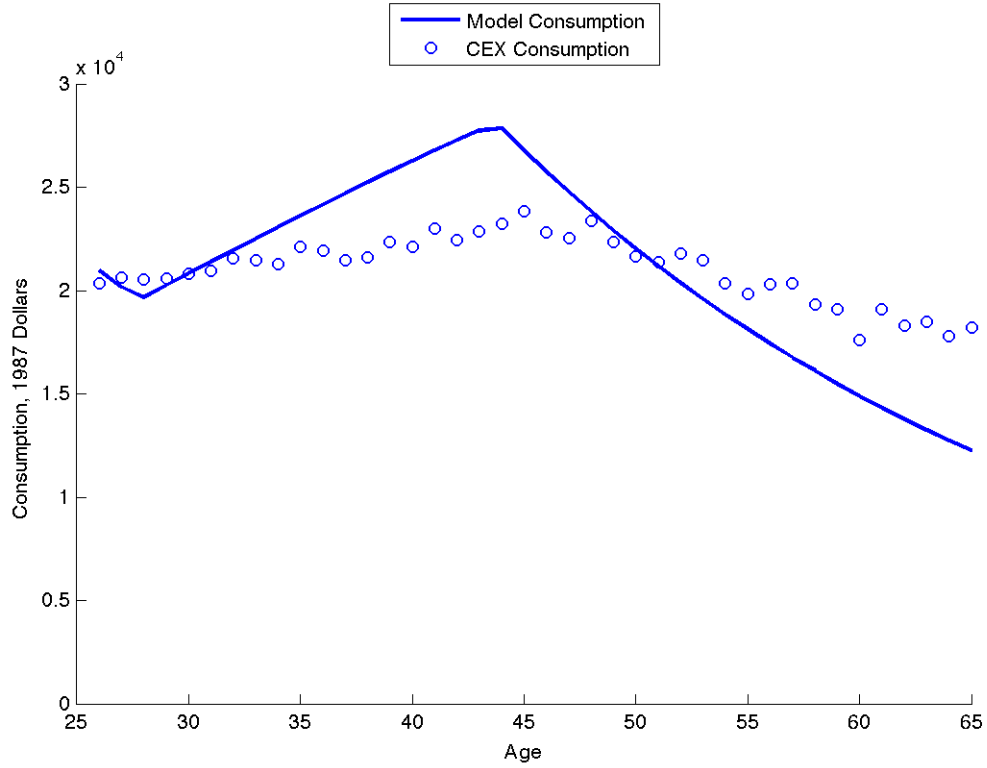
### K.2 Firms

Intermediate goods firms face the same profit maximization and production function as before. However, now all borrowing of capital (paying the rental rate  $r^k$ ) must be intermediated through a bank as well. Firms rent capital services at rate  $\rho_b$ , and individuals can lend capital at rate  $\rho_l$ . The arbitrage equation for the real interest rate is now given by:

$$1 + r_{t,b} = \frac{r_{t,b}^k + (1 - \delta) \xi_t}{\xi_{t-1}} \quad (\text{A.88})$$

With a similar arbitrage equation for the interest rate for lending.

**Figure A.2:** Consumption profile in stationary equilibrium



### K.3 General Equilibrium

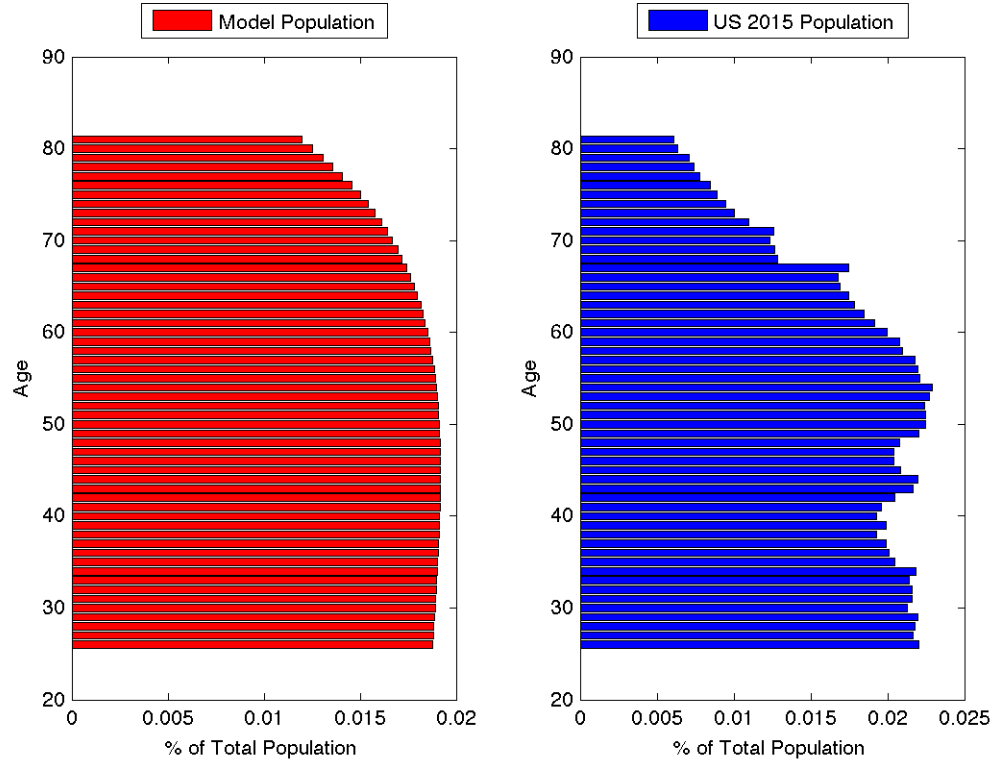
In general equilibrium, the supply for loanable funds must equal the demand:

$$S(r_l) = D(r_b) \quad (\text{A.89})$$

## L Supplemental Figures and Tables

We perform several robustness exercises. The first alternate specification holds all variables at the levels in the main specification, however sets depreciation to be 8%. The second alternate specification holds all variables at the level in the main specification, but sets the utility elasticity of substitution equal to 1. The third alternate specification holds all parameters at the level as in the main specification, but sets the production elasticity equal to 1 (Cobb-Douglas). The fourth alternate specification sets the intertemporal elasticity of substitution to 0.5.

**Figure A.3:** Population pyramid in stationary equilibrium



**Table A.3:** 2015 steady state results, alternate specification 1

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-1.47%	-1.47%
Investment to output ratio	15.9%	15.9%
Consumer debt to output ratio	6.3%	6.3%
Labor share	65.99%	65.99%
Bequests to output	3%	3%



**Table A.4:** Decomposition, alternate specification 1

Forcing variable	$\Delta$ in $r$	% of total $\Delta$
Total interest rate change	-3.60%	100%
Mortality rate	-1.28	34%
Total fertility rate	-1.19	32%
Productivity growth	-1.67	45%
Government debt (% of GDP)	+1.13	-30%
Labor share	-.45	12%
Relative price of investment goods	-0.36	10%
Change in debt limit	+.08	-2%

**Table A.5:** Raising the rate of interest to 1%, alternate specification 1

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	5.40
Government debt (% of GDP)	118%	285%
Productivity growth	0.65%	2.65%
Relative price of investment goods	1.00	2.22

**Table A.6:** 2015 steady state results, alternate specification 2

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-1.47%	-1.47%
Investment to output ratio	15.9%	15.9%
Consumer debt to output ratio	6.3%	6.3%
Labor share	65.99%	65.99%
Bequests to output	3.01%	3%

**Table A.7:** Decomposition, alternate specification 2

Forcing variable	$\Delta$ in $r$	% of total $\Delta$
Total interest rate change	-2.86%	100%
Mortality rate	-1.22	43%
Total fertility rate	-1.36	47%
Productivity growth	-1.28	45%
Government debt (% of GDP)	+1.54	-54%
Labor share	-.33	12%
Relative price of investment goods	-0.30	10%
Change in debt limit	+.08	-3%

**Table A.8:** Raising the rate of interest to 1%, alternate specification 2

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	4.35
Government debt (% of GDP)	118%	279%
Productivity growth	0.65%	3.29%
Relative price of investment goods	1.00	1.78

**Table A.9:** 2015 steady state results, alternate specification 3

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-.81%	-1.47%
Investment to output ratio	18.03%	15.9%
Consumer debt to output ratio	6.3%	6.3%
Labor share	65.40%	65.99%
Bequests to output	3%	3%

**Table A.10:** Decomposition, alternate specification 3

Forcing variable	$\Delta$ in $r$	% of total $\Delta$
Total interest rate change	-2.85%	100%
Mortality rate	-1.45	46%
Total fertility rate	-1.41	45%
Productivity growth	-1.70	54%
Government debt (% of GDP)	+1.58	-50%
Labor share	-.3	10%
Relative price of investment goods	0	0%
Change in debt limit	+.13	-4%

**Table A.11:** Raising the rate of interest to 1%, alternate specification 3

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	3.18
Government debt (% of GDP)	118%	208%
Productivity growth	0.65%	2.11%
Relative price of investment goods	1.00	≠

**Table A.12:** 2015 steady state results, alternate specification 4

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-1.43%	-1.47%
Investment to output ratio	15.9%	15.9%
Consumer debt to output ratio	6.3%	6.3%
Labor share	65.99%	65.99%
Bequests to output	2.34%	3%

**Table A.13:** Decomposition, alternate specification 4

Forcing variable	$\Delta$ in $r$	% of total $\Delta$
Total interest rate change	-7.07%	100%
Mortality rate	-3.36	38%
Total fertility rate	-3.10	35%
Productivity growth	-4.0	45%
Government debt (% of GDP)	+3.27	-37%
Labor share	-1.11	13%
Relative price of investment goods	-0.83	9%
Change in debt limit	+.29	-3%

**Table A.14:** Raising the rate of interest to 1%, alternate specification 4

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	2.56
Government debt (% of GDP)	118%	163%
Productivity growth	0.65%	1.51%
Relative price of investment goods	1.00	1.95

## M Changes in the US Economy since 1970

Our choice of 1970 as a starting point of our analysis is motivated primarily by the fact that there have been substantial changes in the fundamentals of the US economy since the 1970s. The baby boom began soon after World War II, increasing fertility from 2.5 children per women to a peak of 3.62 children per women in 1960.<sup>74</sup> Although the baby boom started in 1945, the children of the baby boom did not enter economic maturity (age 26) until 1970, thus we are able to look at the impact of the baby boom on the evolution of interest rates in our model starting in 1970. Mortality has also changed substantially since 1970. In 1970 the life expectancy at birth was 70.8 years old, growing to 78.7 in 2010. The combination of a slower fertility rate and a decrease in mortality leads to the aging population we see today. This data on fertility and mortality are the raw data we put into the model. For every year from 1970 to 2015, we input the exact total fertility rate from UN fertility data, along with the exact age-specific mortality rate from CDC life tables. After 2015, we assume we are in a steady state with respect to birth rates and mortality.

<sup>74</sup> After the well known “baby boom” came the less well known “baby bust”, in which total fertility in the US has gradually fallen to 1.88 children per women, below the replacement rate, before recovering roughly to the replacement rate seen today

Gordon (2016) argues that productivity growth in the future will likely stagnate, and that long run growth rates might be half of the 1.9 percent annual rate from 1960-2007. This prediction does not rely on a forecast of a future growth slowdown, for over the past four decades productivity growth has already slowed; since 1970s worker productivity has been slower by .8 percentage points per year than the eight decades before. We incorporate this by using estimated series for productivity growth rates from Fernald (2012). The exact productivity data is used as input in the model from 1970's to present day, while from 2015 onwards we hold it constant at 0.65.

Other important factors driving the real interest rate is the relative price of investment goods, the labor share of output, and government debt. Greenwood, Hercowitz and Krusell (1997) and Fernald (2012) have documented that the relative price of investment goods has fallen by 30% since 1970. Since the relative price of investment goods directly affects capital accumulations, this has the potential to have a substantial effect on real interest rates. We set the relative price of capital goods in our model to match the decline since the 1970s. Karabarbounis and Neiman (2014) and Elsby, Hobijn and Şahin (2013) provide data that the labor share has declined substantially since 1970. We model the declining labor share in our model very simply, through an increase in the profit share  $\frac{1}{\theta_t}$ . We choose the path for  $\theta_t$  to match the decline in the labor share from Elsby, Hobijn and Şahin (2013). Government debt is also an important factor that has changed over the time period. Since 1970, total government debt (including state and local debt) has increased to 118% of GDP. We set government debt to exactly match the data over the time period.

## N Literature Estimates

**Table A.15:** Literature Estimates of Parameters, Part A

<i>Parameter</i>	<i>Source</i>
<b>Beta</b>	
0.988-0.990	Kaplan (2012)
0.956-0.960	Gourinchas and Parker (2002)
1.011	Ríos-Rull (1996)
0.92	Krueger and Kubler (2006)
0.96-1.04	Constantinides, Donaldson and Mehra (2002)
<b>Intertemporal Elasticity of Substitution</b>	
0.54-0.62	Kaplan (2012)
0.71-2.00	Gourinchas and Parker (2002)
0.40	Glover et al. (2011)
0.25-1.00	Ríos-Rull (1996)
0.50	Krueger and Kubler (2006)
0.167-0.250	Constantinides, Donaldson and Mehra (2002)
<b>Borrowing Limit (Various)</b>	
0.18 (Debt to GDP)	Guerrieri and Lorenzoni (2011)
30% Average Wage Income	Kaplan (2012)
0	Gourinchas and Parker (2002)
80% of the value of risky assets they own	Hur (2016)
No borrowing limit	Ríos-Rull (1996)
No borrowing limit	Krueger and Kubler (2006)
0	Constantinides, Donaldson and Mehra (2002)
20% of value of home	Ríos-Rull and Sánchez-Marcos (2008)
0	Aiyagari (1994)
0	Krusell and Smith (1998)
25% of lifetime wage	Iacoviello and Pavan (2013)
<b>Capital Share</b>	
0.36	Ríos-Rull (1996)
0.25	Auerbach and Kotlikoff (1987)
0.33	Mankiw, Romer and Weil (1992)
0.30	Krueger and Kubler (2006)
0.30-0.34	Constantinides, Donaldson and Mehra (2002)

**Table A.16:** Literature Estimates of Parameters, Part B

<i>Parameter</i>	<i>Source</i>
<b>Depreciation</b>	
0.10-0.15	Jorgenson (1996)
0.05	Ríos-Rull (1996)
0.06-0.12	Nadiri and Prucha (1996)
<b>Production Elasticity of Substitution</b>	
0.67-0.70	Oberfield and Raval (2014)
1.00	Berndt (1976)
0.40-0.90	Antras (2004)
0.50-0.60	Klump, McAdam and Willman (2007)
1.25	Karabarbounis and Neiman (2014)
1.30-1.60	Piketty (2014)
0.84	Herrendorf, Herrington and Valentinyi (2015)
0.78	Alvarez-Cuadrado, Van Long and Poschke (2014)
<b>Retailer Elasticity of Substitution</b>	
6.00	Christiano, Eichenbaum and Evans (2005)
6.00	Galí (2015)
7.87	Woodford (2003)
4.00	Nakamura and Steinsson (2009)
4.00	Berry, Levinsohn and Pakes (1995)
3.00	Midrigan (2011)
7.00	Golosov and Lucas (2007)
<b>Bequest Parameter</b>	
Based on bequests as percent of permanent income	Altig et al. (2001)
Based on a wealth transfer share of 60%	De Nardi (2004)