Searching for Neo-Fisher: A Model of Animal Spirit Driven Recessions*

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Abstract

This paper develops a model of recessions caused by fluctuations in optimism or pessimism about future output, i.e. animal spirits. The key assumption is that beliefs are not model-consistent, leading to optimal Keynesian policy — fiscal and monetary expansion. If feedback between current and expected income is too high, optimal policies reverse in sign, and Anti-Keynesian/Neo-Fisherian policies are optimal, in line with previous literature with rational expectations. The relationship between the states is characterized by a simple diagram, a general equilibrium Keynesian cross, the slope of which reflects the feedback between current and expected income. If feedback is low and the slope of the cross is less than 1, normal Keynesian results hold: greater than 1, Neo-Fisherian results are ascendant.

Keywords: Self fulfilling expectations, non-rational expectations, liquidity trap, zero lower bound, Neo-Fisherian policies.

JEL Classification: E12, E21, E52

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1 Introduction

This paper builds a model of recessions caused by low expectations about future output and employment. When consumers are pessimistic about the future they cut spending, leading to involuntary unemployment. Optimal policy is Keynesian: fiscal and monetary expansion. The key differentiating feature of the model is that beliefs are not model-consistent ('rational'), which leads to policy conclusions which are diametrically opposite to an important recent class of similar models with rational expectations. In Schmitt-Grohé and Uribe (2017), Heathcote and Perri (2018), and Mertens and Ravn (2014), there are sunspot recessions caused by self-fulfilling low expectations, however optimal policy is "Anti-Keynesian" (AK): higher interest rates and lower government spending raise output. In addition, an interest-rate peg stabilizes the economy, a "Neo-Fisherian" (NF) result.

The model introduces a framework for analyzing belief-driven recessions without rational expectations. Expectations are formed in reference to past and present economic variables, with a completely general functional form. The setting is a 3 period overlapping generations model as in Eggertsson, Mehrotra and Robbins (2019), which allows for closed form solutions. Agents are permanent income consumers, subject to a borrowing limit; firms are monopolistically competitive. Monetary policy is given by a Taylor rule, and there is a Phillips curve relation between inflation and unemployment.

The model shows that under low feedback between current and expected future income, a pessimistic expectations shock leads to a decline in aggregate demand and results in involuntary unemployment. Under these conditions, optimal policy is Keynesian: increased government spending and lower interest rates boost output. Conversely, if feedback becomes too strong, AK and NF properties emerge: a pessimistic expectation shock *raises* output, as does a cut in government spending or an increase in nominal interest rates. In addition, an interest-rate peg stabilizes the economy at full employment.

The boundary between Keynesian and AK/NF states is characterized by a simple diagram, an expectations-augmented general equilibrium Keynesian cross, shown in figure 1. The diagram shows aggregate spending as a function of income. The difference from the standard Keynesian cross is that interest rates and *expectations of output* are a function of current income. If there is excessive feedback between current and expected output, the slope of the GE expenditure function can exceed unity, intersecting the 45 degree line from below (panel b). This ensures that any impulse that increases spending in partial-equilibrium leads to a decrease in income and spending in general equilibrium. This characterization highlights that AK/NF states only occur in a sense when the economy is too responsive to changes in income.

The model's dynamic behavior is likewise dictated by the slope of the expectations augmented GE cross. With low feedback and a slope less than unity, under a Taylor-rule the economy converges to full employment. With a slope of

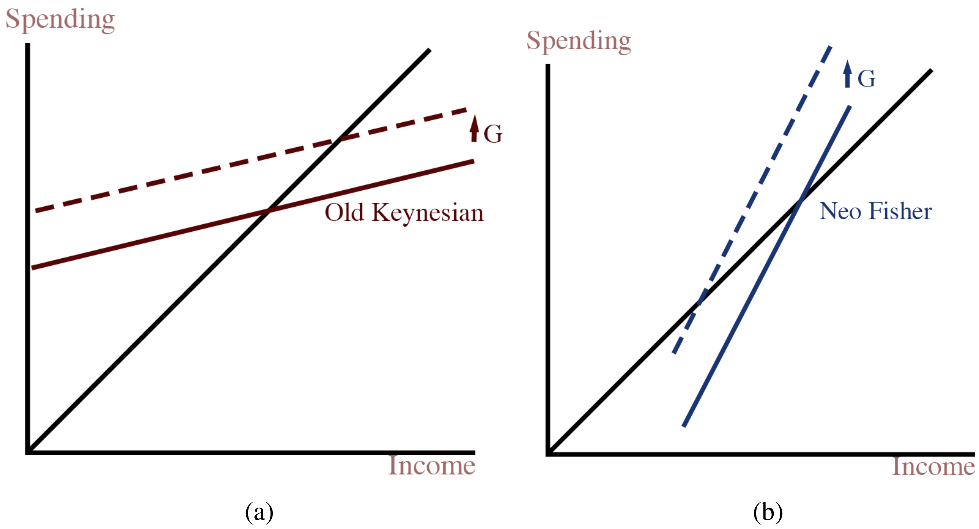


Figure 1: General equilibrium Keynesian cross (a) Old Keynesian properties (b) Anti-Keynesian / Neo-Fisherian properties

greater than one, the economy converges to a low-employment steady state. In addition, with a slope greater than one, the economy converges to full employment under an interest rate peg.

This paper identifies several peculiar properties of models with AK/NF states. First, such models require implausibly large shifts in expectations in response to changes in government spending or interest rates. Second, AK/NF properties are present only after a type of singularity point in which the government spending multiplier jumps from ∞ to $-\infty$. Finally, positive expectation shocks counterintuitively lead to lower equilibrium expectations, due to the reduction in equilibrium output that flows to negative animal spirits.

1.1 Literature

Our model has three novel features: (i) a general specification of non-rational expectations that produces belief-driven recessions (ii) a characterization of the boundary between AK/NF states as a function of the slope of the GE Keynesian cross (iii) an OLG framework that provides stark analytic comparative statics and dynamics. A distinguishing feature of the model is that belief driven recessions are not driven by 'sunspots', or coordinated rational expectations equilibria. Rather, agents' expectations are an independent variable in and of themselves that are subject to fundamental shocks, which in turn can cause either recessions or booms.

Models with belief driven recessions based on sunspots can be separated into two categories. The first focuses on the existence of multiple RE sunspot equilibria under a Taylor rule and the existence of the ZLB; this includes Benhabib, Schmitt-Grohé and Uribe (2001), Schmitt-Grohé and Uribe (2017), and Heathcote and Perri (2018). A second approach relies on a sunspot equilibria based upon a probability of transitioning from a low employment/inflation economy to a high employment targeted steady state, as in Mertens and Ravn (2014) and Borağan Aruoba, Cuba-Borda and Schorfheide (2018).

The paper is related to the adaptive learning literature, which has studied extensively how learning affects New Keynesian models, and conclusively shown that in most cases AK/NF results do not hold under adaptive learning (Benhabib, Evans and Honkapohja (2014), Christiano, Eichenbaum and Johannsen (2018), Evans and McGough (2018)). This paper extends these results to show that AK/NF result can reemerge if there is a high degree of feedback between spending and income. In doing so, we show for the first time a form of non-rational expectations that generates AK/NF properties. The paper is also closely related to Evans, Honkapohja and Mitra (2016), which studies the effects of pessimistic expectation shocks under adaptive learning. Two differences in the paper are our more general expectation formulation, as well as the OLG setup of this paper.

Farmer and Plotnikov (2010) and Farmer (2013) model depressions caused by self fulfilling beliefs about the stock market and the economy. A key difference with this paper is that in Farmer there is no monetary authority that cuts interest rates if output is below potential. This addition restricts the number of long run equilibria in our model.

A sub-theme in recent macroeconomic literature is the strong feedback mechanisms in New-Keynesian models, which causes surprising results such as the "forward guidance puzzle" and Neo-Fisherian results (see, for example, McKay, Nakamura and Steinsson (2016), Gabaix (2016), and Bilbiie (2018*b*)). These papers find that dampening the feedback mechanisms can help get rid of some of the unrealistic properties. A unique aspect of this paper is the characterization of the feedback process in terms of the GE Keynesian Cross. This gives a very intuitive way of understanding why high feedback leads to AK/NF states.

2 Model with non-rational expectations

Individuals live for three periods: young (y), middle aged (m), and old (o). Agents choose consumption to maximize lifetime utility. Young individuals are constrained by a debt limit, which is a function of the total output of the economy and the interest rate: $C_t^y \leq D * Y_t/\widehat{R_t}$, where D is the debt limit parameter, $\widehat{R_t}$ the subjective expected real interest rate. To simplify the dynamics, we assume that D is low enough that the constraint is always binding: $C_t^y = D * Y_t/\widehat{R_t}$. In the second and third period of life individuals are unconstrained, and maximize utility

$$U_t = \log C_t^m + \beta \log \widehat{C_{t+1}^o}.$$

The only uncertainty for agents is the level of output next period, Y_{t+1}, and the gross inflation rate next period π_{t+1}, which in turn affects the expected real interest rate $\widehat{R_t}$. Agents form expectations $\widehat{Y_{t+1}}$ and $\widehat{\pi_{t+1}}$ for output and prices. We assume that agents act as if $\widehat{Y_{t+1}}$ and $\widehat{\pi_{t+1}}$ will occur with certainty. This assumption is made for simplicity, since the focus of our analysis will be the level, and not the variance, of expectations.

Young agents do not work. The middle aged generation inelastically supply γ units of labor, and old individuals supply $(1-\gamma)$ units of labor. Under full employment, total labor supplied is equal to 1. If there is unemployment, with labor demand $L_t < 1$, individuals are off their labor supply curves, and we assume that the relative fraction of labor supplied by the different generations is unchanged. The middle generation supplies γL_t units of labor, and the old generation $(1-\gamma)L_t$ units.

As will be shown, monopolistic firms will generate profits Π_t. Profits are distributed to the middle and old generation in proportion to their labor earnings: $\Pi_t^m = \gamma\Pi_t$, $\Pi_t^o = (1-\gamma)\Pi_t$. Labor is the only factor of production, and thus total income and output in the economy is given by $Y_t = w_t L_t + \Pi_t$. The budget constraint in middle and old age can then be written as

$$C_t^m + B_{t+1}^o/\widehat{R_t} = \gamma w_t L_t + \gamma \Pi_t + B_t^m = \gamma Y_t + B_t^m \tag{1}$$

$$C_t^o = (1-\gamma)w_t L_t + (1-\gamma)\Pi_t + B_t^o = (1-\gamma)Y_t + B_t^o, \tag{2}$$

where B_{t+1} is the quantity of real bonds purchased at time t, at a price $1/\widehat{R_t}$.

Given the constrained maximization problem, optimal consumption is

$$C_t^y = D * Y_t/\widehat{R_t}$$

$$C_t^m = \frac{1}{1+\beta}(\gamma Y_t + B_t^m + (1-\gamma)\widehat{Y_{t+1}}/\widehat{R_t})$$

$$C_t^o = B_t^o + (1-\gamma)Y_t.$$

2.1 Subjective Expectations

Agents form expectations of future output subjectively, and are not necessarily model consistent. Expectations are a function of past and present values of any economic variable, as well as past values of expectations and an expectations shock ϵ_t^e:

$$\widehat{Y_{t+1}} = \mathcal{E}^Y(Y_t, Y_{t-1}, \widehat{Y}_t, \pi_t, R_t, ...) + \epsilon_t^e \tag{3}$$

This specification is general enough to encompass a variety of particular expectation formations: adaptive learning,[1] extrapolative expectations,[2] myopic expectations, or even some rational expectation equilibria. A key characteristic of expectation formation will be the degree of feedback between current output and expected future output, $\frac{d\widehat{Y_{t+1}}}{dY_t} \equiv d_1$.

[1]Following Evans, Honkapohja and Mitra (2022), adaptive learning in this setup takes the form $\widehat{Y_{t+1}} = \widehat{Y}_t + \lambda(Y_{t-1} - \widehat{Y}_t)$, where λ is the "gain" parameter.

[2]Given by $\widehat{Y_{t+1}} = Y_t + \xi(Y_t - Y_{t-1})$.

Inflation expectations are likewise a function of past and present economic variables,

$$\widehat{\pi_{t+1}} = \mathcal{E}^{\pi}(Y_t, Y_{t-1}, \widehat{Y}_t, \pi_t, R_t, ...) \tag{4}$$

For the dynamic analysis, we assume that agents form inflation expectations as a function of current inflation and output:

$$\widehat{\pi_{t+1}} = \mathcal{E}^{\pi}(\pi_t, Y_t). \tag{5}$$

This functional form greatly simplifies the dynamical system since there is no state variable for $\widehat{\pi}_t$.

2.2 Monetary Policy

Monetary policy is set via a Taylor rule, with nominal rates a function of the ratio of actual inflation to target inflation π^*,

$$1 + i_t = max(1, (1 + i^*)(\frac{\pi_t}{\pi^*})^{\phi_\pi}) + \epsilon_t^i.$$

Here ϵ_t^i is the monetary policy shock. We assume the Taylor principle is satisfied, and $\phi_\pi > 1$. The central bank's nominal interest target is denoted $(1 + i^*)$, and is set to the full employment real interest rate R^* times the inflation target: $(1 + i^*) = R^* \cdot \pi^*$.

Given this policy rule, when inflation is below target, initially the real interest rate declines when inflation declines, as the central bank lowers the nominal interest rate more than one-to-one with inflation. At the ZLB, however, the nominal interest rate cannot decline further, and then lower inflation leads to higher real interest rates.

2.3 IS-Exp curve

The aggregate expenditure function is the sum of spending by all generations:

$$Spend_t(Y_t) = \nu Y_{t-1} + \overline{MPC}(\widehat{R_t})Y_t + \chi(\widehat{R_t})\widehat{Y_{t+1}} \tag{6}$$

Here ν[3] represents the propensity to consume out of past income (which affects spending through the accumulation of assets), $\overline{MPC}(\widehat{R_t})$ the propensity to consume out of current income, and $\chi(\widehat{R_t})$[4] the propensity to consume out of expected future income. $\overline{MPC}(\widehat{R_t})$ is a weighted average of the individual generations' marginal propensities to consume out of current income. The weights are the relative income shares of each generation.[5]

[3] $\nu = \frac{\beta D}{1+\beta}$.

[4] $\chi(\widehat{R_t}) = mpc^m(1-\gamma)/\widehat{R_t}$

[5] $\overline{MPC}(\widehat{R_t}) = mpc^y \frac{D}{\widehat{R_t}} + mpc^m \gamma + mpc^o(1-\gamma)$.The weights sum to more than one because the young generation borrows. The marginal propensities to consume of the young, middle aged, and old are 1, $\frac{1}{1+\beta}$, and 1, respectively.

In equilibrium aggregate spending equals aggregate income. Solving for Y_t yields the Investment-Savings-Expectations (IS-Exp) curve.[6]

$$Y_t = \alpha(\widehat{R_t}) \left(\nu Y_{t-1} + \chi(\widehat{R_t}) \widehat{Y_{t+1}} \right) \tag{7}$$

The partial equilibrium Keynesian multiplier, $\alpha(\widehat{R_t})$,[7] equals $\frac{1}{1-\overline{MPC}(\widehat{R_t})}$.

Figure 2b, blue line, shows the short run IS-Exp curve in inflation-output space.[8] The curve displays a negative relationship between real interest rates and output. With a Taylor rule, real rates decrease when inflation is below target, increasing employment and output — thus above the kink, the blue line slopes downward. After the ZLB binds, lower inflation leads to higher real rates, lowering employment — thus the blue line slopes upwards below the kink.

2.4 Aggregate supply

2.4.1 Final goods sector

There is a unit mass of monopolistically competitive final goods firms that produce differentiated goods with labor. The final good composite is the CES aggregate of these differentiated final goods, which are indexed by i:

$$Y_t = \left[\int_0^1 y_t^f(i)^{\frac{\Lambda_t - 1}{\Lambda_t}} di \right]^{\frac{\Lambda_t}{\Lambda_t - 1}}.$$

Final goods firms set prices in each period, and face a demand curve of the form $y_t^f(i) = Y_t^D \left(\frac{p_t(i)}{P_t} \right)^{-\Lambda}$, where Y_t^D is aggregate demand for the composite good from consumers, P_t is the nominal price index of the final good aggregate, and Λ is a measure of a firm's market power.[9]

Each firm uses labor to produce output according to a linear technological function $y_t(i) = L(i)$. Firms choose real prices $\frac{p_t(i)}{P_t}$ and quantities $y_t^f(i)$ to maximize real profits, subject to the production constraint. The marginal cost of producing a unit of final good is the real wage w_t. They thus maximize

$$\Pi_t(i) = \frac{p_t(i)}{P_t} Y_t^D \left(\frac{p_t(i)}{P_t} \right)^{-\Lambda} - w_t Y_t^D \left(\frac{p_t(i)}{P_t} \right)^{-\Lambda} \tag{8}$$

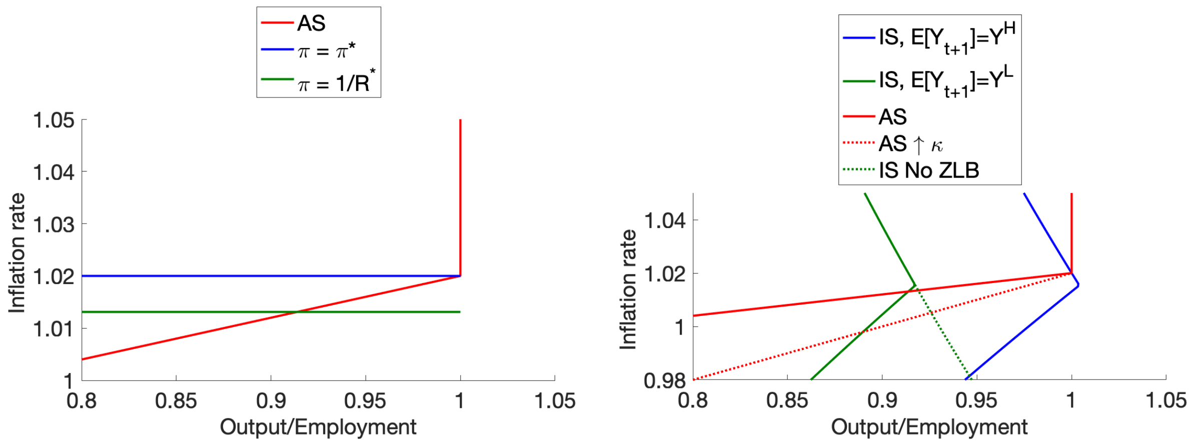
The pricing optimality condition is a markup $\mu \equiv \frac{\Lambda}{\Lambda - 1}$ over marginal cost (which is the real wage): $\frac{p_t(i)}{P_t} = \mu w_t$. Since the wage is constant across all firms, each firm make the same pricing decision, and thus $p_t(i) = P_t$, yielding $w_t = \frac{1}{\mu}$.

[6] This equation uses the fact that in equilibrium, total borrowing of the young generation must equal lending from the middle ($B_t^m = -DY_t = -B_t^o$).

[7] Exact formula $\alpha(\widehat{R_t}) = \frac{1}{\left[1 - mpc^y \frac{D}{R_t} - mpc^m \gamma - mpc^o (1-\gamma)\right]}$.

[8] We use the following calibration for this exercise: $\beta = .29$, $\gamma = .9$, $D = .1$, $\pi^* = 1.02$, $\phi_\pi = 1.5$, $\kappa = .08$.

[9] The price index for the final aggregate is given by $P_t = \left(\int_0^1 p_t(i)^{1-\Lambda_t} di \right)^{\frac{1}{1-\Lambda_t}}$.



(a) (b)

Figure 2: (a) long run aggregate supply and IS-Exp curve (b) short run IS-Exp curves under difference expectations for future output $\widehat{Y_{t+1}}$.

2.4.2 Inflation

Inflation is determined through a non-forward looking Phillips curve relationship between inflation and unemployment:

$$(\pi_t - \pi^*) = \kappa(Y_t - Y^*) \tag{9}$$

The non-forward looking Phillips curve is equivalent to the relationship between inflation and unemployment in Eggertsson, Mehrotra, and Robbins (2019), Schmitt-Grohé and Uribe (2017), Bilbiie, Monacelli and Perotti (2019), and Bilbiie (2018*b*), and microfounded in Bilbiie (2018*a*).

2.5 Temporary Equilibrium

The economy reduces to a six equation system:

$$\begin{aligned}
Y_t &= \alpha(\widehat{R_t})\left(\nu Y_{t-1} + \chi(\widehat{R_t})\widehat{Y_{t+1}}\right) \\
1 + i_t &= max(1, (1+i^*)(\frac{\pi_t}{\pi^*})^{\phi_\pi}) + \epsilon_t^i \\
\widehat{R_t} &= (1+i_t)/\widehat{\pi_{t+1}} \\
\pi_t &= \pi^* - \kappa(1 - Y_t) \\
\widehat{\pi_{t+1}} &= \mathcal{E}^\pi(Y_t, Y_{t-1}, \widehat{Y}_t, \pi_t, R_t, ...) \\
\widehat{Y_{t+1}} &= \mathcal{E}^Y(Y_t, Y_{t-1}, \widehat{Y}_t, \pi_t, R_t, ...) + \epsilon_t^e
\end{aligned}$$

The first three equations are combined to form the IS-Exp curve in $\pi - Y$ space, the fourth equation is the AS curve, and the fifth and sixth determine the dynamics of expectation formation. A "temporary" equilibrium, in the Hicks (1939) sense, for the economy occurs at the intersection of the AS and IS-Exp curve. At this point, given the current state of expectations, inflation and output are set at the level which sets aggregate supply equal to aggregate demand. Outside the steady state, this is depicted in figure 2b, solid blue line.

In a steady state, output and expectations of output are constant and identical. Setting these equal in equation 7 yields the unique equilibrium real interest rate $R^* = \frac{D(1+\beta)+(1-\gamma)}{\beta(\gamma-D)}$.

There are an infinite number of combinations of nominal interest rates and inflation rates that achieve R^*. However, given the Taylor rule of the monetary authority, there are only two ways for the nominal interest rate to be constant in a steady state. Either inflation is on target at $\pi^H \equiv \pi^*$, or the ZLB is binding, and the inflation rate consistent with $i = 0$ and R^* is equal to the inverse of the natural rate, $\pi^L \equiv 1/R^*$.

Figure 2a shows the two long-run IS curves in inflation-output space, one at inflation rate π^H, one at π^L. The two steady state equilibria are the intersections of these long run IS-Exp curves with the AS curve. The twin equilibria are equivalent to those in Schmitt-Grohé and Uribe (2017). We refer to output and inflation in the full employment steady state as Y^H and π^H, and in the low steady state Y^L and π^L.

There is also a third steady state, a "zero" steady state, where all production and employment ceases: we denote this by Y^0. As will be seen, if expectations drop low enough the economy can asymptote to this extreme condition.

3 Expectation driven recessions

Negative expectation shocks can drive the economy to recession, as long as there is not too much feedback between income and spending in the economy.

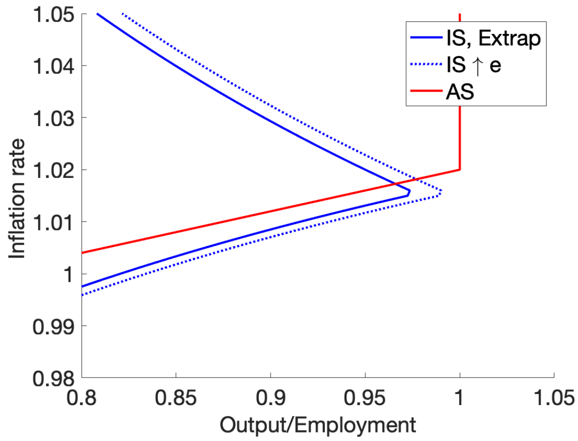
Proposition 1. *If the total derivative of the expenditure function (equation 6) is less than 1 (*$\frac{dSpend_t}{dY_t} < 1$*), then* $\frac{\partial Y_t}{\partial \epsilon_t^e} > 0$*, and a negative expectation shock decreases output.*

Corollary 1. *If the slope of the GE expenditure function is greater than 1, then* $\frac{\partial Y_t}{\partial \epsilon_t^e} < 0$*, and a negative expectations shock increases output.*

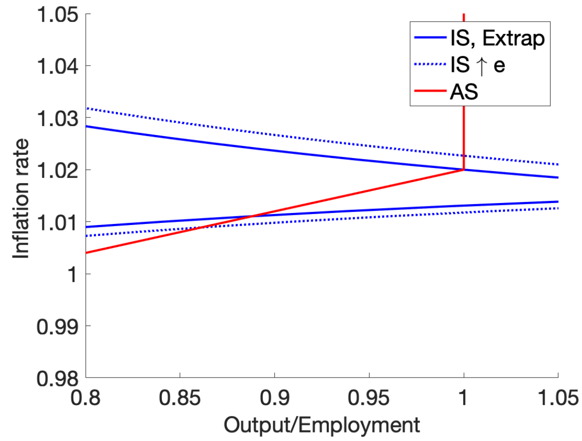
Proof. See appendix B. □

Negative shocks to confidence lead to spending cuts, which cycle back into lower income and further cuts in spending through the standard multiplier process. The condition on the slope of the Keynesian cross ensures that there is not too much feedback between income and spending.

Figure 2b illustrates proposition 1. The blue line is the IS-Exp under high expectations, however a negative confidence shock shifts the curve to the left, lowering output and inflation, shown by the green line.



(a)



(b)

Figure 3: IS-Exp curves, extrapolative expectations (a) low feedback $\xi = .1$ (b) high feedback $\xi = .25$.

The slope of the GE Keynesian cross is given by

$$\frac{d\ Spend_t}{dY_t} = \overline{MPC}(\widehat{R_t}(Y_t)) + \chi(\widehat{R_t})\frac{d\mathcal{E}^Y}{dY_t} + \frac{\partial \overline{MPC}}{\partial \widehat{R_t}}\frac{\partial \widehat{R_t}}{\partial Y_t}Y_t$$
$$-\widehat{R_t}^{-1}\chi(\widehat{R_t})\widehat{Y_{t+1}}\frac{\partial \widehat{R_t}}{\partial Y_t} \overset{?}{<} 1 \qquad (10)$$

When income increases, spending changes from the partial equilibrium effect (the first term), through expectations (the second term), as well as two channels that flow from interest rate movements (third and fourth terms). First, increases in the interest rate lower the amount the younger generation can borrow, lowering the $\overline{MPC}$. Second, higher interest rates lower lower consumption of the middle agent through substitution effects. If the economy is away from the ZLB, then $\frac{\partial \widehat{R_t}}{\partial Y_t} > 0$, and thus both of the final terms will be negative.

An important determinant of the slope of the GE cross is the degree feedback between current income and expected future income, $\frac{d\mathcal{E}^Y}{dY_t}$. Too much feedback raises the slope of the GE Keynesian cross above 1, reversing the effects of an expectation shock, the first "anti-Keynesian" result. This can be illustrated by considering an agent with extrapolative expectations, $\widehat{Y_{t+1}} = Y_t + \xi(Y_t - Y_{t-1}) + \epsilon_t^e$. If ξ is low, as depicted in figure 3a ($\xi = .1$), there is a single intersection of the IS-Exp curve and AS curve, and a positive expectations shock increases output. With higher feedback ($\xi = .25$), shown in figure 3b, there may be multiple equilibria. At the low employment equilibrium, the slope of the GE Keynesian cross is greater than 1, and the same expectation shock leads to a decline in output.

If nominal prices are more flexible, this will increase the steepness of the AS curve. As seen in figure 2b, red dashed line, this will only decrease employment and output in the recession.

Proposition 1 holds even if there were no zero-lower-bound constraint. This can be seen by eliminating the kink in the IS-Exp curve, shown in figure 2b, green dashed line: a negative shock to expectations will still shift the IS-exp curve to the left, leading to lower employment.

4 Fiscal and monetary policy

The government spends G_t, and raises revenue through taxes T_t and issuing government bonds B_t^G. The government budget constraint is given by

$$B_{t+1}^G / \widehat{R_t} = B_t^G + G_t - T_t.$$

We focus on a fiscal policy in which a fraction of taxes T_t is paid by each generation in lump sum to balance the budget: τ_t^y, τ_t^m, and τ_t^o. For analytic simplicity, we assume the government does not tax the old: $\tau_t^o = 0$.[10]

The IS-Exp Curve then becomes a function of government policy:

$$Y_t = \alpha(\widehat{R_t}) \Big(\nu Y_{t-1} + \chi(\widehat{R_t}) \widehat{Y_{t+1}} + \\ \Theta(\tau_t) G_t - \Upsilon(\tau_t) B_t^G + \Upsilon(\tau_t) \frac{B_{t+1}^G}{\widehat{R_t}} \Big), \tag{11}$$

where $\Upsilon(\tau_t) = \tau_t^y + \frac{1}{1+\beta}\tau_t^m$, $\Theta(\tau_t) = 1 - \tau_t^y - \frac{1}{1+\beta}\tau_t^m$.

The same condition that governs the response of the economy to expectation shocks in proposition 1 determines the effects of tax-financed fiscal spending:

Proposition 2. *If*

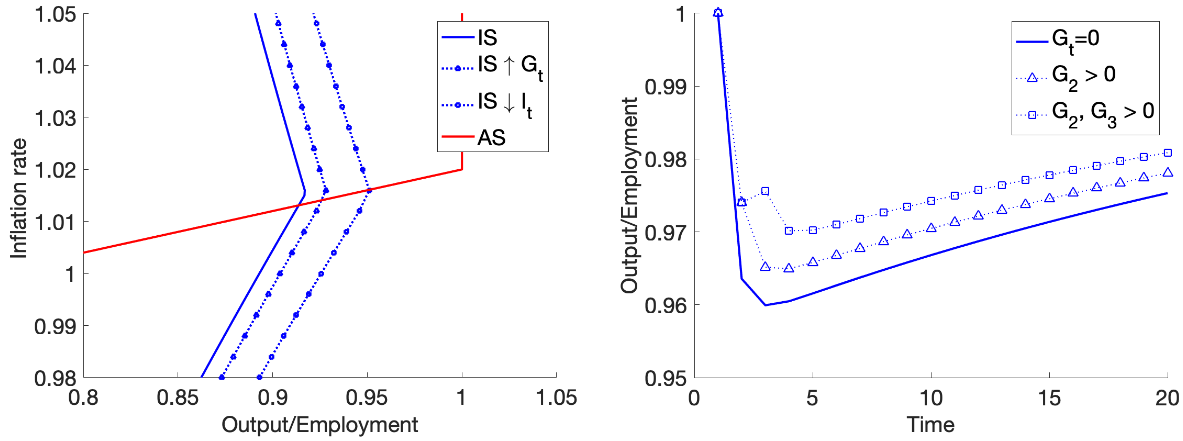
- *The total derivative of the expenditure function (equation 6) is less than 1:* $\frac{dSpend_t}{dY_t} < 1$
- *The increase in government spending is tax financed, i.e. there is no change in* B_{t+1}^G

then $\frac{\partial Y_t}{\partial G_t} > 0$*, and an increase in government spending boosts output.*

Proof. See appendix B. □

Corollary 2. *If the slope of the GE expenditure function is greater than 1, then* $\frac{\partial Y_t}{\partial G_t} < 0$.

[10] This assumption simplifies the dynamics. It is not a completely innocuous assumption: it cuts down the amount of Ricardian equivalence in the model, since for the middle generation, an increase in future taxes no longer depresses spending.



(a) (b)

Figure 4: (a) IS-exp curves, with and without tax financed government spending (b) Transition path, adaptive learning, $\lambda = .4$.

Figure 4a shows the results of an increase in government spending, which shifts the IS curve to the right, boosting output. The overall spending multiplier depends on (i) the partial equilibrium multiplier $\alpha(\widehat{R_t})$ (ii) $\Theta(\tau_t)$, the distribution of taxes across generations (iii) B^G_{t+1} and B^G_t, the amount of spending which is financed by borrowing. The greater the proportion of taxation that falls on the middle generation and proportion of the spending that is debt-financed, the larger the overall multiplier.

4.1 Monetary policy shock

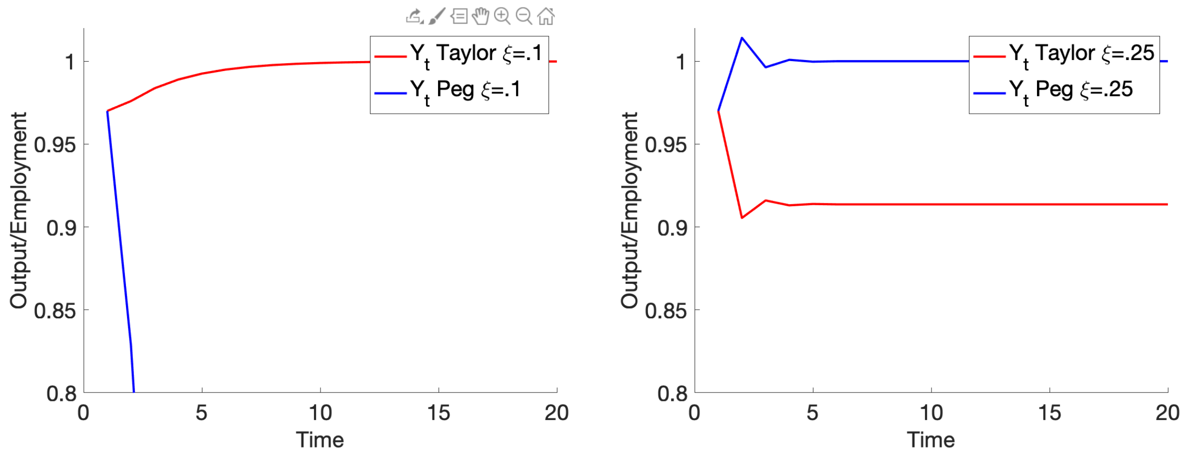
A negative interest rate shock ϵ^i_t functions similarly to an increase in government spending: the IS-Exp curve is shifted to the right, increasing output as long as the GE Keynesian cross slope is less than unity. This is seen in figure 4a, triangle markers.

Proposition 3. *If the total derivative of the expenditure function (equation 6) is less than 1, then* $\frac{\partial Y_t}{\partial \epsilon^i_t} < 0$.

Proof. See appendix B. □

4.2 Interest rate peg

We now consider the effects of pegging the nominal interest rate, $I_t = R^* \cdot \pi^*$. The same condition that governs the comparative statics above also determines the dynamics under a peg: if the slope of the GE expenditure function is less than unity, a peg destabilizes the full employment steady state, and causes output to fall to Y^0. If the slope is greater than one, the model displays NF properties, and a peg stabilizes the economy around Y^H.



(a) (b)

Figure 5: Dynamics of Extrapolative expectations, Taylor rules and interest rate pegs (a) transition paths with low feedback and no NF properties, $\xi = .1$ (b) transition paths with high feedback and NF properties, $\xi = .25$

Figure 5b (blue line) shows transition paths under an interest rate peg with extrapolative expectations and high feedback, $\xi = .25$, with GE expenditure slope greater than 1. With high feedback the economy is NF, and output converges to Y^H. If the monetary authority instead follows a Taylor rule (red line), the economy does not tend towards full employment, but instead asymptotes to Y^L.

Why does a peg ensure convergence to full employment? Proposition 3 showed that with the MPS is greater than 1, the normal signs of fiscal and monetary policy are reversed. When this occurs, a rise in real interest rates, which occurs under a peg, raises output, expectations of output, and output growth, stabilizing the economy around full employment.

Figure 5a (blue line) shows transition dynamics under a peg with low expectations feedback, $\xi = .1$. With the GE expenditure slope less than 1, there are no longer NF properties, and an interest rate peg causes the economy to tend towards Y^0. The reason why Y^H is unstable is that with output below full employment, an interest rate peg always raises real interest rates relative to a Taylor rule. This shifts the IS-Exp curve to the left, leading to a decline in output.

To formally prove stability/instability under a peg, the functional form of expectations must be specified; we do so here for extrapolative expectations and adaptive learning.

Proposition 4. *If*

- *Expectations of output are either:*
  - *Extrapolative:* $\widehat{Y_{t+1}} = Y_t + \xi(Y_t - Y_{t-1}) + \epsilon_t^e$
  - *Adaptive learning:* $\widehat{Y_{t+1}} = \widehat{Y}_t + \lambda(Y_{t-1} - \widehat{Y}_t) + \epsilon_t^e$

- *The total derivative of the expenditure function (equation 6) is less than 1:* $\frac{dSpend_t}{dY_t} < 1$

- *Interest rates are pegged* $I_t = I^*$

then Y^H *is an unstable steady state.*

Proof. See appendix B. □

4.3 Fiscal Policy in a Mertens & Ravn economy

We now specify expectations modeled in the same form as Mertens and Ravn (2014). We assume agents are hit with a negative expectation shock, which causes output to be unexpectedly low ($Y_t = Y_t^U$). Output in the future follows a Markov process. With probability z output will continue at the same low level $Y_{t+1} = Y_t^U$. With probability $(1 - z)$, GDP will recover, and the economy will return to full employment. A crucial factor in this expectation formation is that the actual level of output in the economy at period t is endogenous, which in turn means expected output in period $t + 1$ is endogenous.

The expenditure function is given by

$$\begin{aligned} Spend_t^{MR}(Y_t^U) &= \nu Y_{t-1} + [\overline{MPC}(\widehat{R_t}) + z\chi(\widehat{R_t})]Y_t^U + \\ &(1 - z)\chi(\widehat{R_t})Y^H + \Theta(\tau_t)G_t - \Upsilon(\tau_t)B_t^G + \Upsilon(\tau_t)\frac{B_{t+1}^G}{\widehat{R_t}} \end{aligned} \tag{12}$$

The marginal propensity to spend out of current income is proportional to z, the probability that output will remain low. If z is high enough, the slope of the expenditure function will rise above one, leading as usual to AK properties.

Proposition 5. *If*

- *Expected future output will continue at the current low level with probability* z *and return to full employment with probability* $(1 - z)$

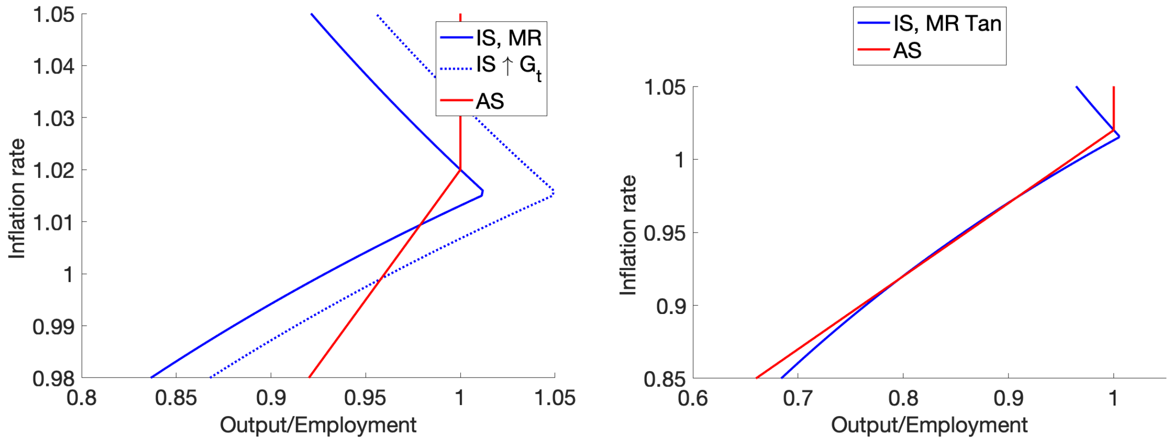
- *The total derivative of the expenditure function (equation 12) is greater than 1:* $\frac{dSpend_t}{dY_t} > 1$

- *The increase in government spending is tax financed, i.e. there is no change in* B_{t+1}^G

then $\frac{\partial Y_t}{\partial G_t} < 0$, *and an increase in government spending decreases output.*

Proof. Follows from proposition 2. □

Figure 6a displays the IS-Exp curve and reveals multiple equilibria where the IS-Exp intersects with the AS curve. The AK properties of the model are present at the low employment equilibrium, where the slope of the expenditure function is greater than 1. An increase in government spending shifts the IS-Exp curve to the right, leading to a *decline in output*.



(a) (b)

Figure 6: Model with Mertens & Ravn expectations (a) IS-Exp: government multipliers are negative but finite (b) IS-Exp: government multipliers are $-\infty$.

5 Dynamics

If a negative expectations shock is small, the system will return to full employment, as long as the slope of the GE Keynesian cross is less than 1. Again, this is the same condition governing the model's comparative statics. With feedback sufficiently low, when the economy is below full employment the lower real interest rates from a Taylor rule increase spending, demand, and employment, ensuring convergence to the 'high' equilibrium.

To formally prove this, the functional form of expectations must be restricted —- we now assume they are a function of only period t and $t-1$ values of any economic variable: output, expectations, inflation, interest rates, etc.

$$\widehat{Y_{t+1}} = \mathcal{E}^Y(Y_t, Y_{t-1}, \widehat{Y}_t, \pi_t, ...) + \epsilon^e_t. \tag{13}$$

To reduce the dimensionality of the system, and focus our attention on output expectations, we assume that inflation expectations are only a function of current output and inflation:

$$\widehat{\pi_{t+1}} = \mathcal{E}^\pi(\pi_t, Y_t). \tag{14}$$

Given this specification, inflation, expected inflation, and interest rates can be written as a function of output, and thus we can write $\widehat{Y_{t+1}} = \mathcal{E}^Y(Y_t, Y_{t-1}, \widehat{Y}_t) + \epsilon^e_t$.[11]

For the economy to converge to full employment, expectations must also satisfy regularity conditions that ensure that close to full employment, expectations

[11] We can eliminate inflation from the equation using the Phillips curve, and eliminate interest rates using the Taylor rule and equation 13.

do not diverge. In the vicinity of Y^H, expectation dynamics are characterized by the total derivative of $\widehat{Y_{t+1}}$ with respect to current output, past output, and past expectations: $\frac{d\mathcal{E}^Y}{dY_t} \equiv d_1$, $\frac{d\mathcal{E}^Y}{dY_{t-1}} \equiv d_2$, $\frac{d\mathcal{E}^Y}{d\widehat{Y_t}} \equiv d_3$.[12] The regularity conditions are, mathematically,

I. $\mathcal{E}^Y(Y^H, Y^H, Y^H) = Y^H$, $\mathcal{E}^\pi(\pi^*, Y^H) = \pi^*$: under the subjective expectations, (Y^H, π^*) is a steady state.

II. (d_1, d_2, d_3) lie underneath surfaces s_1 and s_2 defined by[13]

$$s_1 : d_1 = \frac{1}{\chi}[(1 - d_3) * (-a_{11} - a_{12}) + \chi(1 - d_2 - d_3)]$$
$$s_2 : d_1 = \frac{1}{\chi}[(1 + d_3)(2\nu - a_{11} - a_{12}) + \chi(1 + d_2 + d_3)]$$

For a given functional form of expectations, the economic interpretation of conditions (I-II) is straightforward. For example, if expectations are formed through adaptive learning ($\widehat{Y_{t+1}} = \lambda Y_{t-1} + (1 - \lambda)\widehat{Y_t}$) or adaptive expectations based on current output ($\widehat{Y_{t+1}} = \lambda Y_t + (1 - \lambda)\widehat{Y_t}$), with $0 < \lambda < 1$, they will always be satisfied. For a more general version, with λ unrestricted, they will be satisfied as long as λ is not too large.[14] For extrapolative expectations, $\widehat{Y_{t+1}} = Y_t + \xi(Y_t - Y_{t-1})$, conditions (I)-(II) will hold as long as ξ is not too large.[15]

Proposition 6. *If*

- *The total derivative of the expenditure function (equation 6) is less than 1:* $\frac{dSpend_t}{dY_t} < 1$
- *Regularity conditions (I)-(II) on expectations are satisfied*

then there is a unique saddle path solution to Y^H in the vicinity of Y^H.

Proof. See appendix B. □

Figure 7a shows the effect of a small shock to expectations under adaptive learning, with the GE expenditure slope less than 1. In period 1 the economy is at full employment, but in period two there is a small negative expectation shock, $\epsilon_2^e = -.05$, meaning expected output in period 3 is $\widehat{Y_3} = .95$. The lower expectations lead to a drop in output in period 2 and subsequent periods. The lower expectations are somewhat, but not completely self fulfilling —- actual output $Y_3 \approx .96 > \widehat{Y_3}$.

[12]Given the above discussion, these total derivatives incorporate both feedback from income into output and inflation expectations.

[13]See the proof in appendix B for the definition of parameters a_{11} and a_{12}.

[14]For expectations on past output, the condition is $0 < \lambda < 2 + \frac{2\chi}{2\nu - a_{11} - a_{12}}$. Based on current output, the condition is $0 < \lambda < 1 + \frac{2\nu - a_{11} - a_{12}}{2\nu + 2\chi - a_{11} - a_{12}}$.

[15]$0 < \xi < \frac{2\nu - a_{11} - a_{12}}{2\chi}$

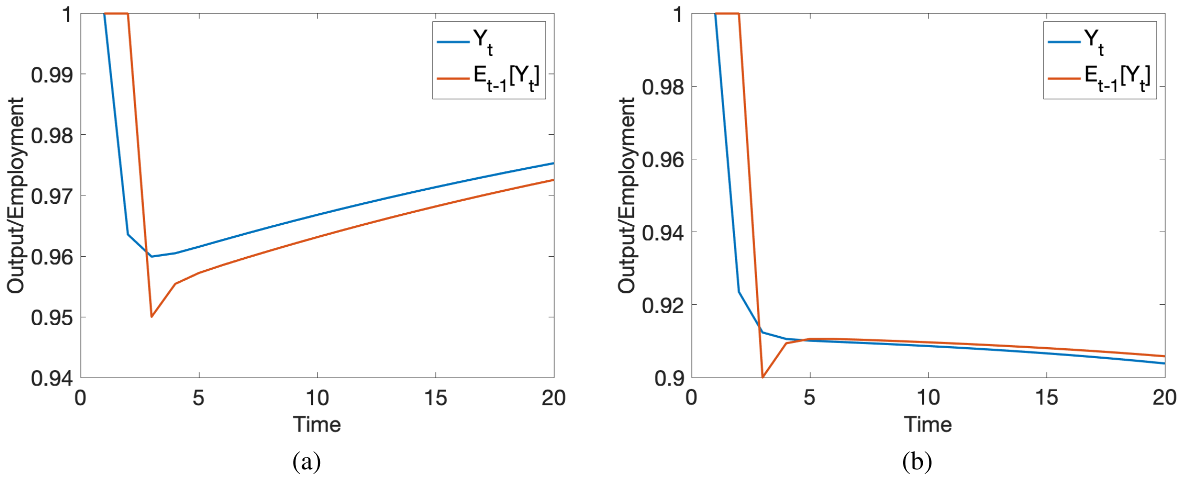


Figure 7: Transition dynamics, adaptive learning, $\lambda = .4$. (a) Small shock, output returns to Y^H(b) Large shock, output goes to Y^0.

Although the system eventually returns to full employment, this may take a considerable amount of time. Lower output in the previous period and lower expectations are significant drags on output. The only thing pulling up the system are lower interest rates, but depending on the elasticity of intertemporal substitution, these may only have small effects.

If a shock to expectations is large enough, however, the system will not return to full employment, but will slowly tend towards Y^0. If $Y < Y^L$, then inflation is low enough such that $R > R^*$. With higher real interest rates, output is depressed even further, which further reduces expectations, cycling back to lower output and inflation.

Figure 7b shows transition dynamics for a large negative shock to expectations under adaptive learning. In this case the shock is large enough that $Y_2 < Y^L$. In period 3, the higher real interest rate depresses output even further, and the system slowly starts to converge towards zero output. A "death spiral" towards Y^0 seems on its face to be an unrealistic property of the model. It should be noted that it generally takes many periods for the economy to converge to zero, and it assumes no other shocks to expectations or government actions. It also assumes that deflation continues, and that expectations continue to degrade. If there is a minimum level of expectations $\widehat{\bar{\bar{Y}}}$ such that $\widehat{Y}_t > \widehat{\bar{\bar{Y}}}$, output would instead converge to some positive value slightly lower than $\widehat{\bar{\bar{Y}}}$. This is the assumption of Evans, Honkapohja and Mitra (2016). Once again, the expectation shock is almost but not quite self-fulfilling, with $Y_3 > \widehat{Y}_3$. After period 4, however, expectations of output are generally above output.

If the slope of the GE expenditure function is greater than 1, Y^H becomes unstable under a Taylor rule — this is depicted in figure 5b (red line) for extrapolative expectations, $\xi = .25$. Instead, the system converges to Y^L, which is a stable equilibrium. The logic behind the stability of Y^L is as follows: under a Taylor rule, with output just below full employment, lower real interest rates cause output to fall (see proposition 3). Output will fall until Y^L is reached; at levels of output below Y^L, since interest rates are at the ZLB, real interest rates increase, stabilizing output (again from proposition 3).

6 Model under rational expectations

Under rational expectations, the model's dynamics closely resemble that of Schmitt-Grohé and Uribe (2017) and Heathcote and Perri (2018), inclusive of Neo-Fisherian results of monetary and fiscal policy.

The first differences under RE is a shift in the dynamical system: now Y_t is a jump variable, and Y_{t-1} is the only state variable. As a result, not only are there multiple steady state equilibria, but there are also multiple solution paths. For any initial state Y_{t-1}, there is a single rational expectations paths to full employment Y^H, and an infinite number of paths that lead to Y^L.

Proposition 7. *Under RE and a Taylor rule with $\phi_\pi > 1$, Y^H is a saddle point, thus starting from any employment rate Y_{t-1} in the vacinity of Y^H there is a unique path to full employment.*

Proof. See appendix B. □

Figure 8a, blue line, shows convergence to full employment for an initial employment rate of $Y_{t-1} = .95$. While there is only a single solution path that leads to Y^H, there are an infinite number of avenues to Y^L.

Proposition 8. *Under RE and a Taylor rule with $\phi > 1$, Y^L is a stable sink: there are an infinite number of solutions that lead from any initial starting condition Y_{t-1} to Y^L.*

Proof. See appendix B. □

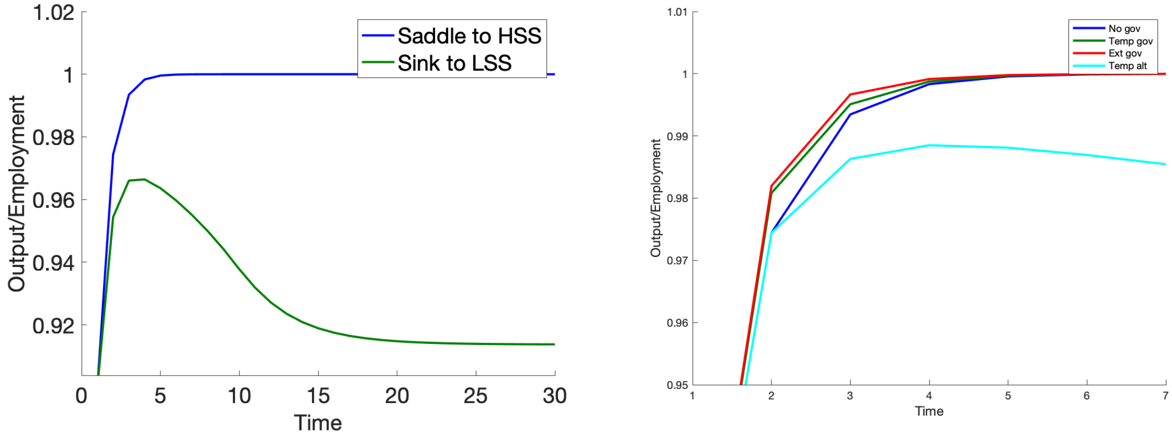
Figure 8a, green line, shows one such path, from a starting value $Y_{t-1} = .95$.

An interest rate peg eliminates the unintended steady state Y^L and ensures convergence to full employment.

Proposition 9. *Under an interest rate peg $I_t = R^* \cdot \pi^*$, there is only a single steady state at full employment. The full employment steady state is a stable sink, with an infinite number of paths converging towards full employment.*

Proof. See appendix B. □

The interest rate peg ensures that the lower steady no longer exists, since in any steady state it must be the case that $\pi = \pi^*$. While this property is somewhat stabilizing, the peg does introduce another property which is somewhat destabilizing: the full employment steady state is no longer a saddle point. There are



(a) (b)

Figure 8: Model with rational expectations. (a) Transition paths: saddle to high steady state, sink to low steady state. (b) Saddle paths to Y^H under government spending.

therefore an infinite number of rational expectation paths that lead away from full employment before eventually returning.

The effects of fiscal policy are complicated by multiple rational expectations equilibria. The one exception to the multiplcitiy is the unique saddle path that exists to Y^H. Along this path there are no AK effects of government spending.

Proposition 10. *Along the saddle path transition to the high steady state, a tax financed increase in government spending increases output on impact:*

$$\frac{\partial Y_t}{\partial G_t} \geq 0$$

Proof. See appendix B. □

Figure 8b, dark blue line, shows the saddle path without government spending, and the green line shows the system with a one time tax-financed increase. The temporary shock to government spending in period t shifts up output and employment along the entire path.

7 Searching for Neo-Fisher

As discussed in the introduction, much of the previous literature on belief-driven recessions under rational expectations displays AK/NF properties. The results of this paper thus far show that AK/NF can re-emerge under non-rational expectations: propositions 1, 2, 3, and 4 together show that the effects of a negative expectation shock, together with whether fiscal and monetary policy displays

AK/NF properties, are jointly determined by a single condition: the slope of the GE expenditure function.

Our results also shed light on *why* existing rational expectation models display AK/NF properties. In Mertens and Ravn (2014), the AK properties are caused by the high feedback between expectations and spending at the low-employment equilibrium. NF properties are caused by two factors. First, the interest rate peg ensures that the low steady state cannot exist, since in any steady state it must be the case that $\pi = \pi^*$. Second, under RE dynamics of output are governed by the Euler equation, through which higher real interest rates raise the growth rate of output. The Euler equation ensures a peg is stabilizing: when output is below target, higher rates from the peg lead to high output growth. As output approaches full employment, higher inflation pushes down real interest rates, equilibrating output at target.

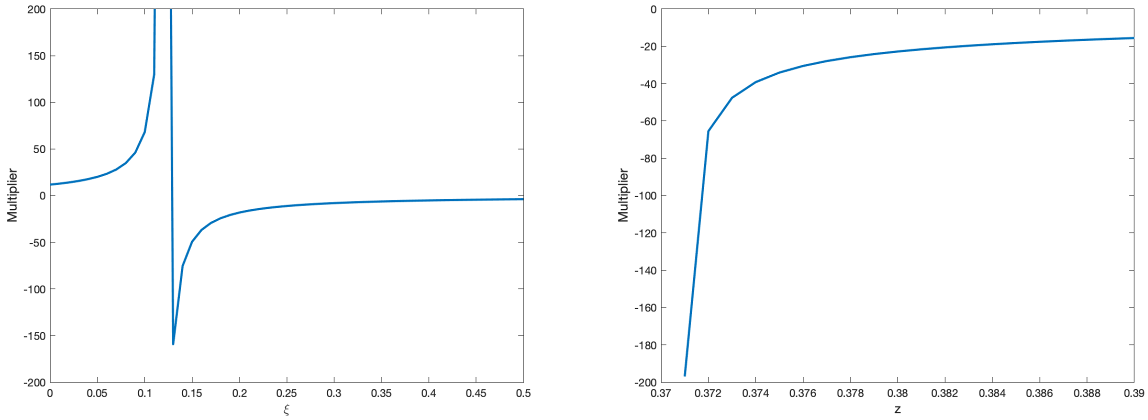
7.1 Properties of AK/NF states

In section 4 it was shown that AK results occur when the general equilibrium MPC is greater than 1. But as the multiplier is equal to $1/(1 - \text{GE MPC})$, the AK region can only begin after a singularity point in which the multiplier is $-\infty$.[16] Under Mertens & Ravens expectations, if z is too small there is no AK equilibrium. But for a z high enough, the multiplier is $-\infty$. As z increases, the magnitude of the multiplier decreases. This is depicted in figure 6b. For z just low enough so that the IS-Exp is tangent to the AS curve, multipliers are negative infinity. As z increases, the magnitude of the (negative) multiplier decreases, as depicted in figure 9b.

Under extrapolative expectations, for ξ small enough, multipliers are positive, and an increase in ξ initially raises the multiplier. As ξ increases the multiplier approaches ∞. At this singularity, the multiplier crosses over to $-\infty$ for the low employment steady state. Figure 9a shows mutlipliers as a function of ξ, and shows the point at which multipliers discontinuously jump from ∞ to $-\infty$.

There is one more perverse property of the model under AK/NF conditions: a positive shock to expectations ϵ_t^e causes a decline in equilibrium expectations $\widehat{Y_{t+1}}$. This follows from proposition 1: the expectation shock decreases output Y_t, which transfers into even lower expectations. In order for the temporary equilibrium to have lower spending, it *must* be the case that expected output decreases from the shock. Under NF conditions, the best way to solve a problem of a low-expectation equilibrium is thus to be hit with a negative expectations shock. This is shown in Appendix B.3 for the case of extrapolative expectations.

[16] Bilbiie (2018*b*) shows a result that is similar in spirit but in a slightly different context, where the region between Neo-Fisherian and Keynesian regions is an inflection point.



(a) (b)

Figure 9: Government multipliers as a function of ξ, extrapolative expectations. (b) multipliers as a function of z, Mertens & Ravn expectations.

8 Discussion and conclusion

The results of this paper show that negative expectations about future output and employment can result in persistent recessions with involuntary unemployment. In such circumstances, traditional Keynesian fiscal and monetary measures can raise output, while Neo-Fisherian interest rate pegs do not result in recovery. A direction implication from the model is the importance of expectations as a distinct force in macroeconomics. The recent research push towards soliciting consumers and business for expectations (see for example, the Survey of Consumer Expectations Armantier et al. (2017) or the work of Coibion, Gorodnichenko and Kumar (2018)) provide evidence of a great diversity of beliefs about inflation, housing prices, and unemployment. More research needs to be done about how these beliefs are formed — for example, in Robert Shiller's recent work on "narrative economics" (Shiller (2017)).

Our findings emphasize the crucial role that feedback plays in shaping the dynamics of output and government policies. The level of feedback in the model determines the impact of negative expectation shocks, whether fiscal/monetary policies are Anti-Keynesian, or if an interest-rate peg displays Neo-Fisherian properties. The level of feedback is determined by the slope of the general-equilibrium expenditure function; if there is excessive feedback and the slope exceeds 1, Anti-Keynesian/Neo-Fisherian outcomes may emerge.

Finally, this study highlights the limitations of the rational-expectations assumption when analyzing belief-driven models of recessions. Anti-Keynesian / Neo-Fisherian results are related to relational expectations due to the strong feedback between income and expectations. However, this paper does show that rational expectations are not a necessary condition for AK/NF results, and that all is needed is a sufficient degree of feedback between income and expectations.

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Online Appendix for

Searching for Neo-Fisher: A Model of Animal Spirit Driven Recessions

Jacob A. Robbins

A Full derivation of model

Consumers maximize

$$U_t = \log C_t^m + \beta \log \widehat{C_{t+1}^o}$$

subject to budget constraints

$$C_t^m + B_{t+1}^o/\widehat{R_t} = \gamma w_t L_t + \gamma \Pi_t + B_t^m = \gamma Y_t + B_t^m \tag{A.1}$$

$$C_t^o = (1-\gamma) w_t L_t + (1-\gamma)\Pi_t + B_t^o = (1-\gamma) Y_t + B_t^o, \tag{A.2}$$

where B_{t+1} is the face value of bonds purchased at time t, at a price $1/\widehat{R_t}$.

The young agent is constrained, and thus $C_t^y = D * Y_t/\widehat{R_t}$. The middle agent is on their Euler equation, yielding the first order condition $\widehat{C_{t+1}^o} = \beta \widehat{R_t} C_t^m$. Plugging this into the budget equation yields the optimal consumption

$$C_t^m = \frac{1}{1+\beta}(\gamma Y_t + B_t^m + (1-\gamma)\widehat{Y_{t+1}}/\widehat{R_t}) \tag{A.3}$$

The old agent simply consumers all of his bonds and assets like there is no tomorrow, and thus

$$C_t^o = (1-\gamma) Y_t + B_t^o \tag{A.4}$$

We can eliminate the asset holdings in all of these optimal consumption equations. In equilibrium, middle age bond holdings are simply what was borrowed when the agent is young, and thus $B_t^m = -D * Y_{t-1}$. In addition, old age bond holdings are simply what was lent when the agent is middle aged. Since the middle aged agent always lends to the young, we have $B_t^o = D * Y_{t-1}$. Finally, we note that the actual returns received by bondholders may differ the realized real interest rate R.

Optimal consumption for each generation is thus

$$C_t^y = D * Y_t/\widehat{R_t}$$

$$C_t^m = \frac{1}{1+\beta}(\gamma Y_t - D * Y_{t-1} + (1-\gamma)\widehat{Y_{t+1}}/\widehat{R_t})$$

$$C_t^o = D * Y_{t-1} + (1-\gamma) Y_t$$

and aggregate consumption is

$$Y_t = C_t^y + C_t^m + C_t^o =$$
$$D * Y_t/\widehat{R_t} + \frac{1}{1+\beta}(\gamma Y_t - D * Y_{t-1} + (1-\gamma)\widehat{Y_{t+1}}/\widehat{R_t}) + D * Y_{t-1} + (1-\gamma)Y_t$$

Grouping terms,

$$Y_t = [\frac{\beta}{1+\beta}D]Y_{t-1} + [D/\widehat{R_t} + \frac{1}{1+\beta}\gamma + (1-\gamma)]Y_t + [\frac{1}{1+\beta}\frac{(1-\gamma)}{\widehat{R_t}}]\widehat{Y_{t+1}}$$

Defining $\nu = \frac{\beta D}{1+\beta}$, $\chi(\widehat{R_t}) = \frac{1}{1+\beta}(1-\gamma)/\widehat{R_t}$, and

$$\overline{MPC}(\widehat{R_t}) = [D/\widehat{R_t} + \frac{1}{1+\beta}\gamma + (1-\gamma)],$$

we have

$$Y_t = \nu Y_{t-1} + \overline{MPC}(\widehat{R_t})Y_t + \chi(\widehat{R_t})\widehat{Y_{t+1}} \tag{A.5}$$

Solving for Y_t yields the Investment-Savings-Expectations (IS-Exp) curve.

$$Y_t = \alpha(\widehat{R_t})\left(\nu Y_{t-1} + \chi(\widehat{R_t})\widehat{Y_{t+1}}\right) \tag{A.6}$$

The partial equilibrium Keynesian multiplier $\alpha(\widehat{R_t}) = \frac{1}{\left[1-\frac{D}{\widehat{R_t}}-\frac{1}{1+\beta}\gamma-(1-\gamma)\right]}$

With government spending and taxation, consumer spending is affected by two contrasting forces: the increase in spending from the federal government, and the decrease in spending from individuals from increased taxation. With taxation, optimal consumption for each generation (assuming no old age taxation) is given by

$$C_t^y = D * Y_t/\widehat{R_t} - T_t^Y$$
$$C_t^m = \frac{1}{1+\beta}(\gamma Y_t - D * Y_{t-1} - T_t^m + (1-\gamma)\widehat{Y_{t+1}}/\widehat{R_t})$$
$$C_t^o = D * Y_{t-1} + (1-\gamma)Y_t$$

We note that the marginal propensities to consume out of past, current, and future income are unchanged. The only differences in the IS-Exp curve will be two additive terms: the direct effects of government spending on consumption, and the indirect effects through taxation.

We focus on a fiscal policy in which a fraction of the total taxes is paid by each generation to balance the budget: τ_t^y and τ_t^m. Given the government budget constraint, the taxes of each generation g equal $T_t^g = \tau_t^g * (B_t^G + G_t - \frac{B_{t+1}^G}{\widehat{R_t}})$. The additional effects of government spending and taxation on the expenditure function is thus

$$G_t - \tau_t^y * (B_t^G + G_t - \frac{B_{t+1}^G}{\widehat{R_t}}) - \frac{1}{1+\beta}\tau_t^m * (B_t^G + G_t - \frac{B_{t+1}^G}{\widehat{R_t}})$$
$$= G_t(1 - \tau_t^y - \frac{1}{1+\beta}\tau_t^m) - B_t^G(\tau_t^y + \frac{1}{1+\beta}\tau_t^m) + \frac{B_{t+1}^G}{\widehat{R_t}}(\tau_t^y + \frac{1}{1+\beta}\tau_t^m)$$
$$= \Theta(\tau_t)G_t - \Upsilon(\tau_t)B_t^G + \Upsilon(\tau_t)B_{t+1}^G/\widehat{R_t}$$

In the last line, we define $\Upsilon(\tau_t) \equiv \tau_t^y + \frac{1}{1+\beta}\tau_t^m$, $\Theta(\tau_t) \equiv 1 - \tau_t^y - \frac{1}{1+\beta}\tau_t^m$.

The IS-Exp curve is then given by

$$Y_t = \alpha(\widehat{R_t})\Big(\nu Y_{t-1} + \chi(\widehat{R_t})\widehat{Y_{t+1}} + \Theta(\tau_t)G_t - \Upsilon(\tau_t)B_t^G + \Upsilon(\tau_t)\frac{B_{t+1}^G}{\widehat{R_t}}\Big) \tag{A.7}$$

In a steady state in which government spending as a fraction of output is given by g and government debt as a fraction of output by b^g, the steady state real interest rate is given by

$$R = \frac{(1-\gamma) + D(1+\beta) + \Upsilon b^g(1+\beta)}{\beta(\gamma - D) + (\Upsilon b^g - \Theta g)(1+\beta)}$$

B Proofs

B.1 Preliminary calculations

The derivative of MPC with respect to $\widehat{R_t}$ is

$$\frac{\partial \overline{MPC}}{\partial \widehat{R_t}} = (-1)D\widehat{R_t}^{-2} < 0$$

The derivative of $\chi(\widehat{R_t})$ with respect to $\widehat{R_t}$ is given by

$$\frac{\partial \chi(\widehat{R_t})}{\partial \widehat{R_t}} = -\chi(\widehat{R_t})/\widehat{R_t}$$

In a steady state, we have

$$Y = \alpha(R^*)\left(\nu Y + \chi(R^*)Y\right) = \alpha(R^*)(\nu + \chi(R^*))Y$$

Therefore we have

$$\alpha(R^*)(\nu + \chi(R^*)) = 1 \tag{A.8}$$
$$1 - \overline{MPC}(R^*) = \nu + \chi(R^*) \tag{A.9}$$

Additionally,

$$\chi(R^*) = \alpha^{-1} - \nu \tag{A.10}$$
$$\nu = \alpha^{-1} - \chi \tag{A.11}$$

From the Fisher equation, we can compute how expected interest rates change in response to output:

$$\widehat{R_t}(Y_t) = \frac{I_t(\pi_t(Y_t))}{\widehat{\pi_{t+1}}(Y_t, \pi_t(Y_t))} \Rightarrow$$
$$\frac{\partial \widehat{R_t}}{\partial Y_t} = \widehat{\pi_{t+1}}^{-1} \frac{\partial I_t}{\partial \pi_t} \frac{\partial \pi_t}{\partial Y_t} - I_t \widehat{\pi_{t+1}}^{-2} [\mathcal{E}_1^\pi + \mathcal{E}_2^\pi \frac{\partial \pi_t}{\partial Y_t}]$$

We compute the pieces separately. From Taylor rules

$$\frac{\partial I_t}{\partial \pi_t} = \begin{cases} \phi_\pi \pi_t^{-1} I_t & \text{if ZLB is not binding} \\ 0 & \text{if ZLB is binding} \end{cases}$$

The Phillips curve is given by $\pi_t = \pi^* - \kappa(1 - Y_t)$, thus $\frac{\partial \pi_t}{\partial Y_t} = \kappa$.

$$\frac{\partial \widehat{R_t}}{\partial Y_t} = \begin{cases} \kappa \phi_\pi \pi_t^{-1} \widehat{R_t} - \widehat{\pi_{t+1}}^{-1} \widehat{R_t} [\mathcal{E}_1^\pi + \mathcal{E}_2^\pi \kappa] \overset{?}{>} 0 & \text{if ZLB is not binding} \\ -\widehat{\pi_{t+1}}^{-1} \widehat{R_t} [\mathcal{E}_1^\pi + \mathcal{E}_2^\pi \kappa] < 0 & \text{if ZLB is binding} \end{cases}$$

Note that evaluated at Y^H, π^H,

$$\frac{\partial \widehat{R_t}}{\partial Y_t} = \pi^{-1} R[\kappa(\phi - \mathcal{E}_2^\pi) - \mathcal{E}_1^\pi]$$

B.2 Propositions

Proof of proposition 1.

Proof. This follows from an analysis of the IS-Exp curve. As long as the GE MPS is less than 1, multipliers will have normal signs. From the IS-Exp we define

$$F(Y_t, \epsilon_t^e) = \nu Y_{t-1} + \overline{MPC}(\widehat{R_t}(Y_t)) Y_t + \chi(\widehat{R_t}) \widehat{Y_{t+1}}(\epsilon_t^e) - Y_t = 0 \tag{A.12}$$

From the implicit function theorem,

$$\frac{\partial Y_t}{\partial \epsilon_t^e} = -\frac{\frac{\partial F}{\partial \epsilon_t^e}}{\frac{\partial F}{\partial Y_t}}$$

For this derivative to be positive, the denominator must be negative, since the numerator is positive:

$$\frac{\partial F}{\partial \epsilon_t^e} = \chi(\widehat{R_t}) > 0$$

The denominator is given by

$$\frac{\partial F}{\partial Y_t} = \underbrace{\frac{\partial \overline{MPC}}{\partial \widehat{R_t}} \frac{\partial \widehat{R_t}}{\partial Y_t} Y_t}_{\equiv a_{11}} + \overline{MPC}(\widehat{R_t}(Y_t)) + \chi(\widehat{R_t}) \frac{d\mathcal{E}^Y}{dY_t} \underbrace{- \widehat{R_t}^{-1} \chi(\widehat{R_t}) \widehat{Y_{t+1}} \frac{\partial \widehat{R_t}}{\partial Y_t}}_{\equiv a_{12}} - 1$$

We note that the condition the denominator is negative is the condition that the general equilibrium Keynesian cross has a slope of less than 1.

The derivative will depend upon whether the ZLB is binding or not. In the case where the ZLB is not binding, feedback between income and spending is lower because the two terms a_{11} and a_{12} are negative. Feedback is increased when the ZLB is binding, as there is an additional feedback mechanism between income and spending. When the ZLB is binding, higher income means higher inflation, which means lower interest rates and higher spending. □

Proof of proposition 2.

Proof. The proof follows similar argument as proposition 1. From the IS-exp equation 11, we define

$$\begin{aligned} F(Y_t, G_t) = \nu Y_{t-1} + \overline{MPC}(\widehat{R_t}(Y_t))Y_t + \chi(\widehat{R_t})\widehat{Y_{t+1}} \\ + \Theta(\tau_t)G_t - \Upsilon(\tau_t)B_t^G + \Upsilon(\tau_t)\frac{B_{t+1}^G}{\widehat{R_t}} - Y_t = 0 \end{aligned} \tag{A.13}$$

By assumption government spending is tax financed, thus there is no change in B_{t+1}^G. In addition, since $\tau_t^m > 0$, $\Theta(\tau_t) > 0$. From the implicit function theorem,

$$\frac{\partial Y_t}{\partial G} = -\frac{\frac{\partial F}{\partial G_t}}{\frac{\partial F}{\partial Y_t}}$$

For this derivative to be positive, the denominator must be negative, since the numerator is positive:

$$\frac{\partial F}{\partial G_t} = \Theta(\tau_t) > 0$$

We will have $\frac{\partial Y_t}{\partial G_t} > 0$ when the GE Keynesian cross has a slope less than one. □

Proof of proposition 3.

Proof. The logic of the proof is similar to proposition 1. We consider the implicit function theorem on the expenditure function:

$$F(Y_t, \epsilon_t^i) = \nu Y_{t-1} + \overline{MPC}(\widehat{R_t}(Y_t, \epsilon_t^i))Y_t + \chi(\widehat{R_t(Y_t, \epsilon_t^i)})\widehat{Y_{t+1}} - Y_t = 0 \tag{A.14}$$

$$\frac{\partial Y_t}{\partial \epsilon_t^i} = -\frac{\frac{\partial F}{\partial \epsilon_t^i}}{\frac{\partial F}{\partial Y_t}}$$

$$\frac{\partial F}{\partial \epsilon_t^i} = \frac{\partial \overline{MPC}}{\partial \widehat{R_t}} \frac{1}{\widehat{\pi_{t+1}}} Y_t + \frac{\partial \chi(\widehat{R_t})}{\partial \widehat{R_t}} \frac{1}{\widehat{\pi_{t+1}}} \widehat{Y_{t+1}} < 0 \tag{A.15}$$

We will thus have $\frac{\partial Y_t}{\partial \epsilon_t^i} < 0$ when the GE Keynesian cross has a slope less than one.

$\square$

Proof of proposition 6.

Proof. From the IS-Exp equation, the dynamical system can be written

$$\begin{aligned} Y_t &= \nu Y_{t-1} + \overline{MPC}(\widehat{R_t}) Y_t + \chi(\widehat{R_t}) \widehat{Y_{t+1}} \\ \widehat{Y_{t+1}} &= \mathcal{E}^Y(Y_t, Y_{t-1}, \widehat{Y_t}, \pi(Y_t), \widehat{\pi_{t+1}}(Y_t, \pi_t)) + \epsilon_t^e \end{aligned}$$

Note that several variables can be eliminated from the analysis: π_t, which through the Phillips curve can be written as a function of Y_t, and $\widehat{\pi_{t+1}}(Y_t, \pi_t)$, which can be written as a function of Y_t and π_t. We can thus write $\widehat{Y_{t+1}} = \mathcal{E}^Y(Y_t, Y_{t-1}, \widehat{Y_t}) + \epsilon_t^e$. We linearize the dynamical system around the high steady state

$$\frac{\partial}{\partial Y_t} = \underbrace{\frac{\partial \overline{MPC}}{\partial \widehat{R_t}} \frac{\partial \widehat{R_t}}{\partial Y_t} Y_t}_{\equiv a_{11}} + \overline{MPC}(\widehat{R_t}(Y_t)) \underbrace{- \widehat{R_t}^{-1} \chi(\widehat{R_t}) \widehat{Y_{t+1}} \frac{\partial \widehat{R_t}}{\partial Y_t}}_{\equiv a_{12}} + \chi(\widehat{R_t}) d_1$$

Where $d_1 \equiv \frac{d\widehat{Y_{t+1}}}{dY_t}$ is the total derivative of expectations with respect to output, taking into account its effect on both inflation and expected inflation. In deviations from steady state (denoted by a tilde) we have

$$\begin{aligned} \tilde{Y}_t = &\left[\overline{MPC}(R^*) + a_{11} + a_{12} + \chi(R^*) d_1\right] \tilde{Y}_t \\ &+ [\nu + \chi(\widehat{R^*}) d_2] \widetilde{Y_{t-1}} + \chi(\widehat{R^*}) d_3 \tilde{\widehat{Y}}_t \end{aligned}$$

Where $d_2 \equiv \frac{d\widehat{Y_{t+1}}}{dY_{t-1}}$, $d_3 \equiv \frac{d\widehat{Y_{t+1}}}{d\widehat{Y_t}}$. We define the GE Keynesian multiplier as

$$a_1 = \left[1 - \left[\overline{MPC}(R^*) + a_{11} + a_{12} + \chi(\widehat{R^*}) d_1\right]\right]^{-1}$$

We note $a_1 > 0$ by the assumption that $\frac{dSpend_t}{dY_t} < 1$. In matrix form, the system is

$$\begin{bmatrix} \widetilde{Y_t} \\ \widetilde{\widehat{Y_{t+1}}} \end{bmatrix} = \begin{bmatrix} (\nu + \chi d_2) a_1 & \chi d_3 a_1 \\ d_2 + d_1(\nu + \chi d_2) a_1 & d_3 + d_1 \chi d_3 a_1 \end{bmatrix} \begin{bmatrix} \widetilde{Y_{t-1}} \\ \tilde{\widehat{Y}}_t \end{bmatrix} \tag{A.16}$$

In order for there to be a unique saddle path, we need both eigenvalues to be less than 1 in absolute value. This condition can be translated into restrictions on the characteristic equation. The characteristic equation is given by

$$\begin{aligned} L(\Theta) = \Theta^2 - \Theta(\nu + \chi d_2)a_1 - \Theta d_3 - \Theta d_1 \chi d_3 a_1 + (\nu + \chi d_2)a_1 d_3 + \\ (\nu + \chi d_2)a_1 d_1 \chi d_3 a_1 - \chi d_3 a_1 d_2 - \chi d_3 a_1 d_1(\nu + \chi d_2)a_1 \\ = \Theta^2 - \Theta(\nu + \chi d_2)a_1 - \Theta d_3 - \Theta d_1 \chi d_3 a_1 + (\nu + \chi d_2)a_1 d_3 - \chi d_3 a_1 d_2 \end{aligned}$$

Since the characteristic equation $L(\cdot)$ is a quadratic equation that is U-shaped ($L'' > 0$), the characteristic equation must satisfy one of three conditions in order for both eigenvalues to be stable:

I. $L(0) \geq 0$, $L'(0) < 0$, $L'(1) > 0$, and $L(1) > 0$

II. $L(0) \geq 0$, $L'(0) > 0$, $L'(-1) < 0$, and $L(-1) > 0$

III. $L(0) < 0$, $L(1) > 0$, and $L(-1) > 0$

We can translate these conditions into restrictions on d_1, d_2, and d_3. The derivative of the characteristic equation, evaluated at $\Theta = 0$, yields

$$\begin{aligned} L'(0) = L'(\Theta) = 2\Theta - (\nu + \chi d_2)a_1 - d_3 - d_1 \chi d_3 a_1 \\ = -(\nu + \chi d_2)a_1 - d_3 - d_1 \chi d_3 a_1 \\ = -a_1[(1 + d_3) * \nu + (d_3 + d_2)\chi + d_3[-a_{11} - a_{12}]] \\ = -a_1[d_3[\nu + \chi - a_{11} - a_{12}] + \nu + d_2 \chi] \\ = -a_1[d_3 a_1 + \nu + d_2 \chi] \end{aligned}$$

At $L(0)$, characteristic equation is

$$L(0) = (\nu + \chi d_2)a_1 d_3 - \chi d_3 a_1 d_2 = \nu a_1 d_3. \tag{A.17}$$

At $L(1)$, the characteristic equation is

$$\begin{aligned} L(1) = 1 - (\nu + \chi d_2)a_1 - d_3 - d_1 \chi d_3 a_1 + (\nu + \chi d_2)a_1 d_3 - \chi d_3 a_1 d_2 \\ = a_1[(1 - d_3)(-a_{11} - a_{12}) + \chi(1 - d_1 - d_2 - d_3)]. \end{aligned}$$

At $L(-1)$, the characteristic equation is

$$\begin{aligned} L(-1) = 1 + (\nu + \chi d_2)a_1 + d_3 + d_1 \chi d_3 a_1 + (\nu + \chi d_2)a_1 d_3 - \chi d_3 a_1 d_2 \\ = a_1[(1 + d_3)(2\nu - a_{11} - a_{12}) - \chi(d_1 - d_2 - d_3 - 1)]. \end{aligned}$$

We can further characterize the conditions for a unique saddle path by considering three lines in d_2 and d_3 space, holding d_1 constant. The first line represents the set of points such that $L(1) = 0$, the second under which $L(-1) = 0$, and the third under which $L'(0) = 0$.

$$\begin{aligned} g_1(d_2) \equiv \frac{[-a_{11} - a_{12} + \chi(1 - d_1)]}{-a_{11} - a_{12} + \chi} - \frac{\chi}{-a_{11} - a_{12} + \chi} d_2 \\ \frac{\partial}{\partial d_3} < 0, \frac{\partial}{\partial d_1} < 0 \end{aligned}$$

The line $g_1(\cdot)$ separates space where $L(1) > 0$ versus $L(1) < 0$. As long as $a_1 > 0$, to the south-west of $g_1(\cdot)$, we have $L(1) > 0$, and to the north-east $L(1) < 0$.

$$\begin{aligned} g_2(d_2) &\equiv \frac{-[2\nu - a_{11} - a_{12} + \chi(1-d_1)]}{2\nu - a_{11} - a_{12} + \chi} - \frac{\chi}{2\nu - a_{11} - a_{12} + \chi} d_2 \\ &= \frac{-[a_1^{-1} + \nu]}{2\nu - a_{11} - a_{12} + \chi} - \frac{\chi}{2\nu - a_{11} - a_{12} + \chi} d_2 \\ & \frac{\partial}{\partial d_3} > 0, \frac{\partial}{\partial d_1} > 0 \end{aligned}$$

The line $g_2(\cdot)$ separates the space where $L(-1) > 0$ versus $L(-1) < 0$. If $a_1 > 0$, to the north-east of $g_2(\cdot)$, we have $L(-1) > 0$, and to the south-west $L(1) < 0$.

$$\begin{aligned} g_3(d_2) \equiv \frac{-\nu}{\nu + \chi - a_{11} - a_{12}} - \frac{\chi}{\nu + \chi - a_{11} - a_{12}} d_2 \\ \frac{\partial}{\partial d_3} < 0, \frac{\partial}{\partial d_1} < 0 \end{aligned}$$

As long as $a_1 > 0$, the line $g_3(\cdot)$ separates the plane into $L'(0) < 0$ to the north-east, versus $L'(0) > 0$ to the south-west. We also define two closely related lines: the $L'(1) = 0$ line, $g_4 = g_3 + 2$, and the $L'(-1) = 0$ line, $g_5 = g_3 - 2$. To the south-west of g_4, $L'(1) > 0$, satisfying part of condition (I), while to the north-east of g_5, $L'(-1) < 0$, satisfying part of condition (II). Between the lines, then, both conditions will be satisfied.

We further characterize the relationship between the lines. First, the slopes are all negative. Second, magnitude of the slope of $g_1(\cdot)$ is larger than the slope of $g_3(\cdot)$, is larger than $g_2(\cdot)$. Third, all intersect at a point $d^* = (d_2^*, d_3^*)$.[17]

[17] To see this, define $y \equiv -a_{11} - a_{12} + \chi$, $w \equiv -a_{11} - a_{12} + \chi + \nu$, $z \equiv 2\nu + \chi - a_{11} - a_{12}$. We then have

$$\begin{aligned} g_1(\cdot) : d_3 &= \frac{y - d_1\chi}{y} - \frac{\chi}{y} d_2 \\ g_2(\cdot) : d_3 &= \frac{-z + d_1\chi}{y} - \frac{\chi}{z} d_2 \\ g_3(\cdot) : d_3 &= \frac{v}{w} - \frac{\chi}{w} d_2 \end{aligned}$$

The intersection of g_1 and g_2 is at

$$g_2 = \frac{yz}{\chi\nu} - \frac{d_1}{\nu}[y + \nu].$$

The intersection of g_1 and g_3 is at

$$\begin{aligned} g_2 &= \frac{y + 2\nu}{\chi\nu} - \frac{d_1}{\nu}[w] \\ &= \frac{yz}{\chi\nu} - \frac{d_1}{\nu}[y + \nu]. \end{aligned}$$

Thus they both intersect at the same point.

Close to d^*, the lines g_1 and g_2 form a cone, bisected by g_3. Figure A.1 displays the lines.

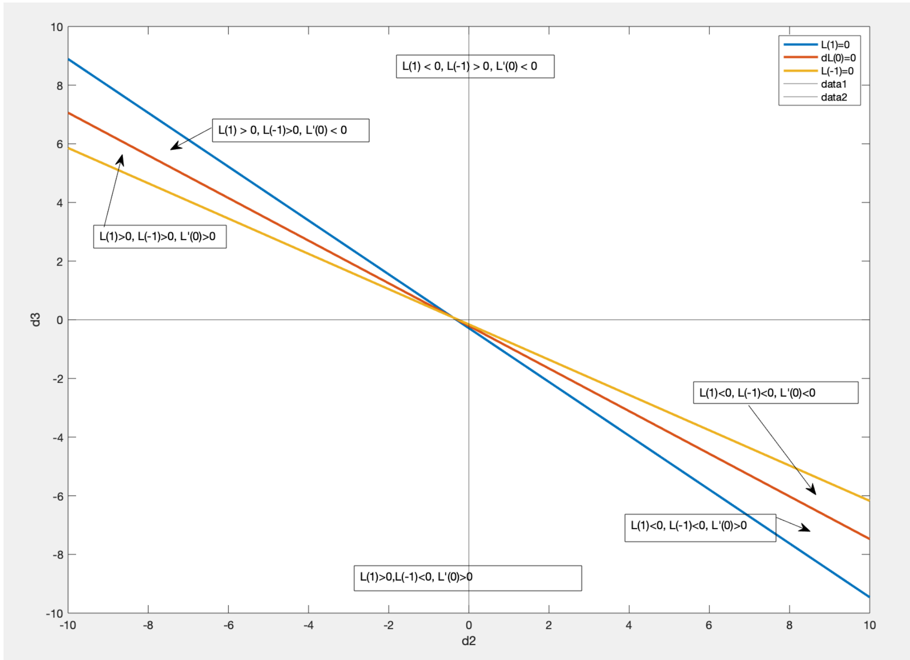


Figure A.1: Characteristic equation lines g_1, g_2, g_3.

Setting aside for now conditions on $L'(1)$ and $L'(-1)$, condition (I) is satisfied in the space between $g_1(\cdot)$ and $g_3(\cdot)$ with $d_3 > 0$. Condition (II) is satisfied in the space between $g_2(\cdot)$ and $g_3(\cdot)$ with $d_3 > 0$; the union of these is the cone between $g_1(\cdot)$ and $g_2(\cdot)$ to the northwest of d^* with $d_3 > 0$. Condition (III) is satisfied in the space between $g_1(\cdot)$ and $g_2(\cdot)$ to the northwest of d^* with $d_3 < 0$. The union of all three conditions is thus the cone formed by $g_1(\cdot)$ and $g_2(\cdot)$ to the north-west of d^*.

Now, taking into account the $L'(1) > 0$ restriction for condition (I), and the $L'(-1) < 0$ for condition (II), these will both be satisfied between the g_4 and g_5 lines, which are, respectively, the g_3 line shifted up and down by two units. The union of all three conditions is thus the cone formed by $g_1(\cdot)$ and $g_2(\cdot)$ to the north-west of d^* that falls within the cone formed by g_4 and g_5. As seen in figure A.2, this restriction bites at extreme values of d_2 and d_3.

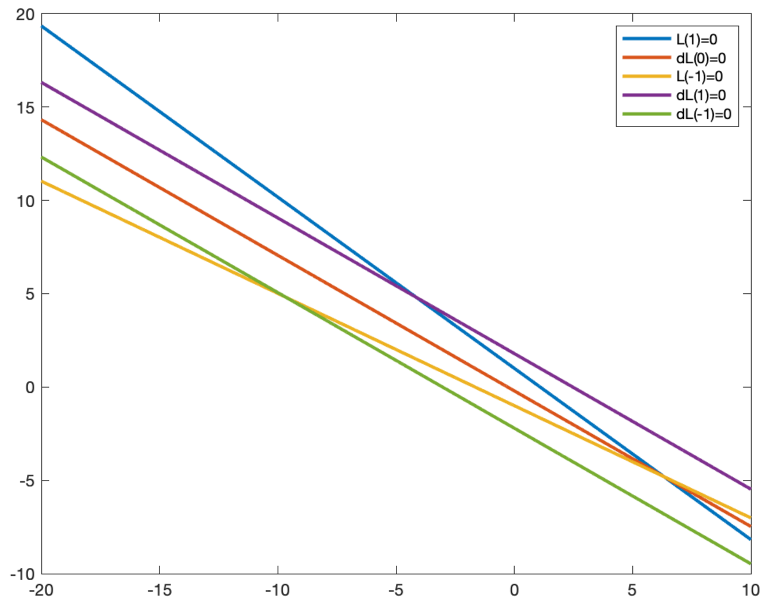


Figure A.2: Additional restrictions of g_4 and g_5 at extreme values of d_2 and d_3.

Now we can begin to discuss how changing d_1 affects the cone. The slope of both $g_1(\cdot)$ and $g_2(\cdot)$ are unaffected by d_1; all that changes is the intercept; increasing d_1 shifts the $g_1(\cdot)$ line down, while shifting the $g_2(\cdot)$ line up. The result is that the intersection point d^* is shifted up and to the left, along the $g_3(\cdot)$ line, meaning that there are fewer points which provide a unique saddle point. This is depicted further in figure A.3, which shows the two surfaces $g_1(\cdot)$ and $g_2(\cdot)$ in 3 dimensions, with the z-axis representing changes in d_1. The points which provide a unique saddle path are the points under both planes, under the 'tent' formed by the surfaces. As d_1 increases, the close to the ceiling of the tent, and the fewer points which converge.

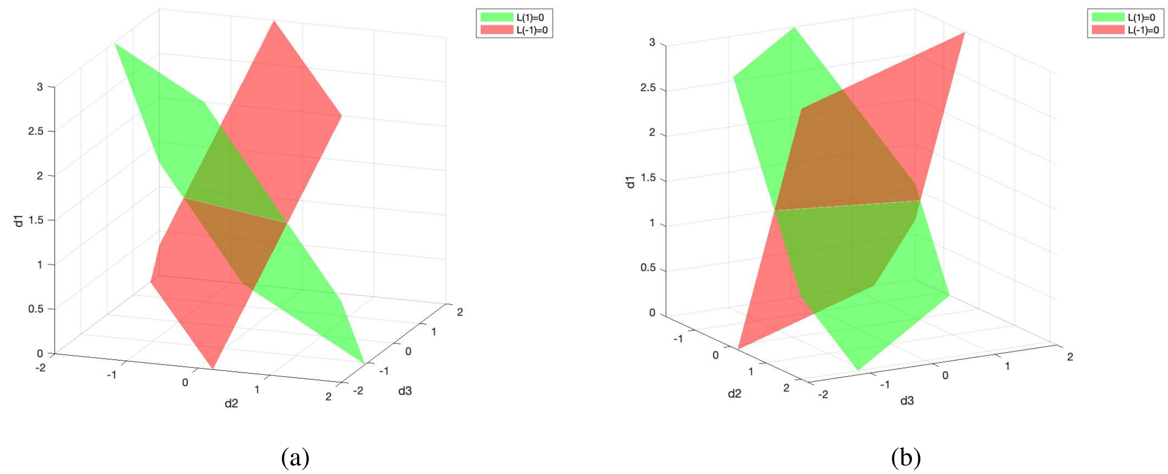


Figure A.3: $L(1) = 0$ and $L(-1) = 0$ planes. The condition for unique saddle path is all points below the $L(1)$ and $L(-1)$ planes.

$\square$

Proof of proposition 4.

Proof. The proof follows a similar line to that of proposition 6. The only difference is that under a peg, $R_t = I^*/\pi_t$, and thus $\frac{\partial \widehat{R_t}}{\partial Y_t} = -I^*\pi_t^{-2}\frac{\partial \pi_t}{\partial Y_t} = -I^*\pi_t^{-2}\kappa < 0$. This means that $a_{11} > 0$ and $a_{12} > 0$.

Under extrapolative expectations, $d_1 = (1+\xi)$, $d_2 = -\xi$, $d_3 = 0$. We then have:

$$\begin{aligned} L(0) &= 0 \\ L'(0) &= -a_1[\nu - \xi\chi] \\ L(1) &= a_1[-a_{11} - a_{12}] \\ L(-1) &= a_1[2\nu - 2\chi\xi - a_{11} - a_{12}] \end{aligned}$$

Since $L(0) = 0$, either case (I) or case (II) is relevant. But case (I) cannot be satisfied, since $L(1) < 0$. In addition, case (II) cannot be satisfied. If $L'(0) > 0$, then $[\nu - \xi\chi] < 0$, which directly implies $L(-1) < 0$. Therefore there is an unstable Eigenvalue.

Under adaptive learning, $d_1 = 0$, $d_2 = \lambda$, $d_3 = 1 - \lambda$.

$$\begin{aligned} L(0) &= \nu a_1(1-\lambda) > 0 \\ L'(0) &= -a_1[(2-\lambda)\nu + \chi + (1-\lambda)(-a_{11} - a_{12})] \\ L(1) &= a_1[\lambda(-a_{11} - a_{12})] < 0 \end{aligned}$$

Since $L(0) > 0$, either case (I) or case (II) is relevant. But note that

$$\begin{aligned} L'(0) &= -a_1[(2-\lambda)\nu + \chi + (1-\lambda)(-a_{11} - a_{12})] \\ &= -a_1[(1-\lambda)[\nu + \chi - a_{11} - a_{12}] + \nu + \lambda\chi] \\ &= -a_1[(1-\lambda)a_1 + \nu + \lambda\chi] < 0 \end{aligned}$$

Since $L'(0) < 0$, $L(1) < 0$, case (I) cannot be satisfied. In addition, since $L'(0) < 0$, case (II) also cannot be satisfied. Therefore there is an unstable Eigenvalue.

$\square$

Proof of Propositions 7.

Proof. Under rational expectations, the dynamical system is given by the IS-Exp curve. We perform a change in variable, defining $Z_{t+1} = Y_t$:

$$Y_t = \nu Z_t + \overline{MPC}(R_t(Y_t, \pi_{t+1}))Y_t + \chi(R_t(Y_t, \pi_{t+1}))Y_{t+1} \tag{A.18}$$

We linearize this around the high steady state.

$$\frac{\partial}{\partial Y_t} = \overline{MPC}(\widehat{R_t}(Y_t)) + a_{11} + a_{12} - 1 = -a_1^{-1}$$

$$\begin{aligned}\frac{\partial}{\partial Y_{t+1}} = \underbrace{\frac{\partial \overline{MPC}}{\partial R_t}\frac{\partial R_t}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial Y_{t+1}}Y_t}_{\equiv b_{11}} + \underbrace{\frac{\partial \chi(R_t)}{\partial R_t}\frac{\partial R_t}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial Y_{t+1}}Y_{t+1}}_{\equiv b_{12}} + \chi(R_t) \\ = [\chi + b_{11} + b_{12}] \equiv b_1\end{aligned}$$

We thus have

$$Y_{t+1} = a_1^{-1}b_1^{-1}Y_t - \nu b_1^{-1}Z_t \tag{A.19}$$

The linearized system in deviations from steady state is given by

$$\begin{bmatrix}\widetilde{Y_{t+1}} \\ \widetilde{Z_{t+1}}\end{bmatrix} = \begin{bmatrix} a_1^{-1}b_1^{-1} & -\nu b_1^{-1} \\ 1 & 0\end{bmatrix}\begin{bmatrix}\widetilde{Y}_t \\ \widetilde{Z}_t\end{bmatrix} \tag{A.20}$$

The Eigenvalues are given by the zeros of the characteristic equation

$$L(\Theta) = \Theta^2 - \Theta a_1^{-1}b_1^{-1} + \nu b_1^{-1} \tag{A.21}$$

Note that $a_1 > 0$ by assumption. In addition,

$$\begin{aligned} b_{11} &= \frac{\partial \overline{MPC}}{\partial R_t}\frac{\partial R_t}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial Y_{t+1}}Y_t \\ &= -DR^{*-2} \times (-1)R^*\pi^{*-1} \times \kappa > 0 \\ b_{12} &= \frac{\partial \chi(R_t)}{\partial R_t}\frac{\partial R_t}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial Y_{t+1}}Y_{t+1} \\ &= -\chi R^{*-1} \times (-1)R^*\pi^{*-1} \times \kappa > 0\end{aligned}$$

We therefore have $L(0) > 0$ and $L'(0) < 0$. In order to have a single unstable eigenvalue for the one non-predetermined variable, the condition for convergence is that $L(1)$ is negative.

$$\begin{aligned} L(1) = 1 - a_1^{-1}b_1^{-1} + \nu b_1^{-1} = \frac{b_1 + \nu - a_1^{-1}}{b_1} \\ = \frac{\nu + \chi - \chi - \nu + a_{11} + a_{12} + b_{11} + b_{12}}{b_1} = \frac{a_{11} + a_{12} + b_{11} + b_{12}}{b_1}\end{aligned}$$

Noting that

$$a_{11} = \frac{\partial \overline{MPC}}{\partial R_t}\frac{\partial R_t}{\partial Y_t}Y_t$$
$$= -DR^{*-2} * \phi_\pi R^* \pi^{*-1}\kappa = -\phi b_{11} < 0$$

$$a_{12} = \frac{\partial \chi(R_t)}{\partial R_t}\frac{\partial R_t}{\partial Y_t}Y_{t+1}$$
$$= -\chi R^{*-1} * \phi_\pi R^* \pi^{*-1}\kappa = -\phi b_{12} < 0$$

$$L(1) = \frac{b_{11}(1-\phi_\pi) + b_{12}(1-\phi_\pi)}{b_1} \tag{A.22}$$

The condition thus boils down to $\phi_\pi > 1$, i.e. that the Taylor Principle is satisfied. □

Proof of Propositions 8.

Proof. The steps are the same as in proposition 7, except now the derivatives are taken around the low steady state. At the low steady state, the real interest rate is unchanged, but the nominal interest rate is 1. From the Fisher equation, we then have $\pi_t = R^{*-1}$. Using the aggregate supply curve, $\pi^L = \left(\pi^* - \kappa(1 - Y^L)\right) = (R^*)^{-1}$. Thus $Y^L = \frac{1}{\kappa}\left((R^*)^{-1} - \pi^L\right) + 1$.

In addition, $a_{11} = 0$, $a_{12} = 0$,

$$b_{11} = \frac{\partial \overline{MPC}}{\partial R_t}\frac{\partial R_t}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial Y_{t+1}}Y_t$$
$$= -DR^{*-2} \times (-1)R^* {\pi^L}^{-1} \times \kappa \times Y^L > 0$$

$$b_{12} = \frac{\partial \chi(R_t)}{\partial R_t}\frac{\partial R_t}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial Y_{t+1}}Y_{t+1}$$
$$= -\chi R^{*-1} \times (-1)R^* {\pi^L}^{-1} \times \kappa \times Y^L > 0$$

We thus have

$$L(1) = \frac{b_{11} + b_{12}}{b_1} > 0$$

There are thus no unstable eigenvalues and one non-predetermined variable, meaning Y^L is a stable sink.

□

Proof of Propositions 9.

Proof. The steps are similar to those in proposition 7. With a peg, $R_t = I^*/\pi_{t+1}$, and thus $\frac{\partial R_t}{\partial Y_t} = 0$. We thus have $a_{11} = 0$, $a_{12} = 0$, $b_{11} > 0$, $b_{12} > 0$, thus $L(1) > 0$ and Y^H is a stable sink.

□

Proof of proposition 10.

The linearized dynamical system under tax-financed government spending is given by

$$Y_{t+1} = a_1^{-1} b_1^{-1} Y_t - \nu b_1^{-1} Z_t - G_t \Theta(\tau_t) b_1^{-1} \tag{A.23}$$

$$Z_{t+1} = Y_t \tag{A.24}$$

Let Y_s^0 be the level of output along the saddle path for the system with zero government spending, and Y_s^{temp} the level of output for the path with a one time increase in government spending in period t. From equation A.23, from period $t+2$ onwards the dynamical system of both paths are the same, since there is no government spending or tax after period t. Only in period $t+1$ is there a difference in the dynamical systems, from the temporary spending. The question of the difference between the two then boils down to: is $Y_t^{temp} > Y_t^0$?

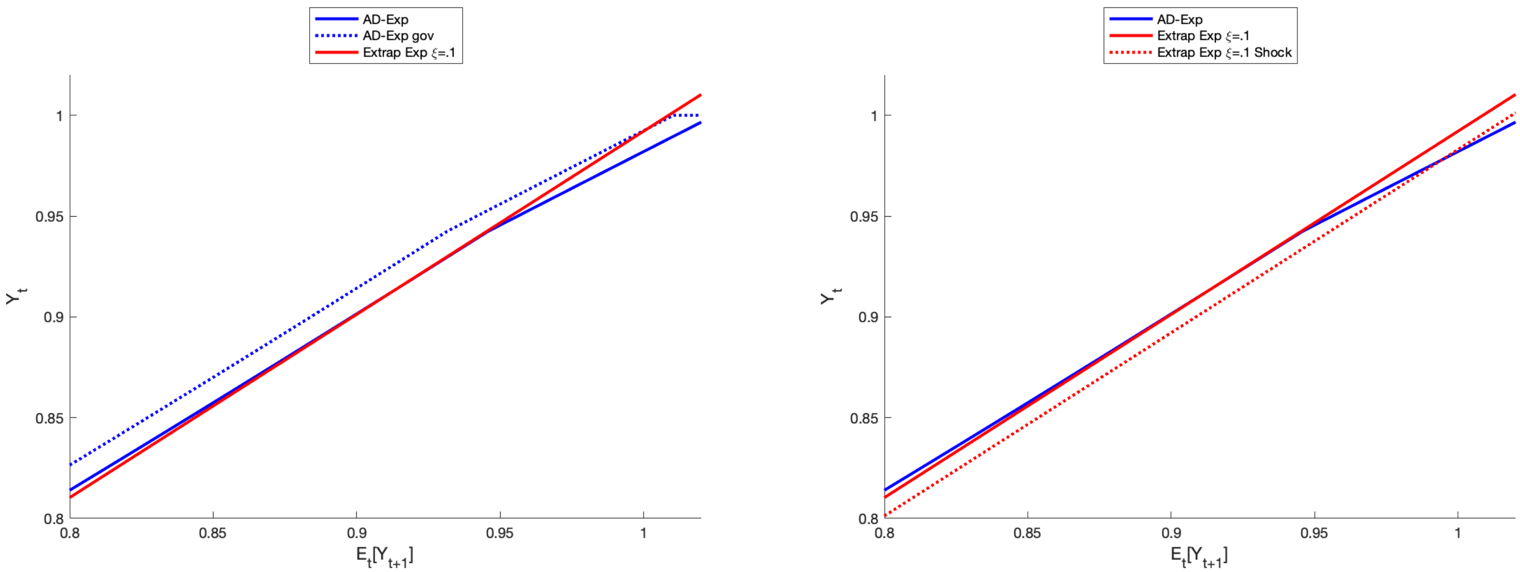
Consider what would happen if $Y_t^{temp} = Y_t^0$. From equation A.23, we would then have $Y_{t+1}^{temp} < Y_{t+1}^0$. But note that this implies that Y_{t+1}^{temp} cannot be on the saddle path to Y^H. If it were, then from the perspective of the time $t+2$ dynamical system, there would be two paths to full employment from starting with initial output Y_t, a contradiction that Y^H is a saddle point.

The transition path where $Y_t^{temp} = Y_t^0$ is shown in figure 8b, teal line. It is thus clear that we must have $Y_t^{temp} > Y_t^0$ on the saddle path to full employment.

B.3 Belief shocks and equilibrium beliefs

In this section we show that under extrapolative expectations, AK/NF properties directly imply that a negative expectations shock ϵ_t^e increases equilibrium expected output $\widehat{Y_{t+1}}$. To do so, we characterize the equilibrium with a change in variables. We first derive an aggregate demand expectations curve (AD-Exp), which is in the $\{Y_t, \widehat{Y_{t+1}}\}$ space. This curve is derived from the IS-Exp curve, equation 11. We fix the level of output Y_t, and then use the IS-Exp curve to find the level of expectation $\widehat{Y_{t+1}}$ consistent with the output level. This gives a curve of equilibrium output Y_t as a function of expectations of future output, $\widehat{Y_{t+1}}$. Figure 2b traces out two points of this aggregate demand curve under low and high expectations. This AD-Exp curve is shown in figure A.4a, blue line.

The second curve is the "belief formation" (Bel) curve, which shows how expectations of output change under different levels of current income, and is derived from the expectation specification $\widehat{Y_{t+1}} = Y_t + \xi(Y_t - Y_{t-1}) + \epsilon_t^e$. Under extrapolative expectations, the slope of the curve is $\frac{1}{1+\xi}$. This is displayed by the red line in figure A.4a.



(a) (b)

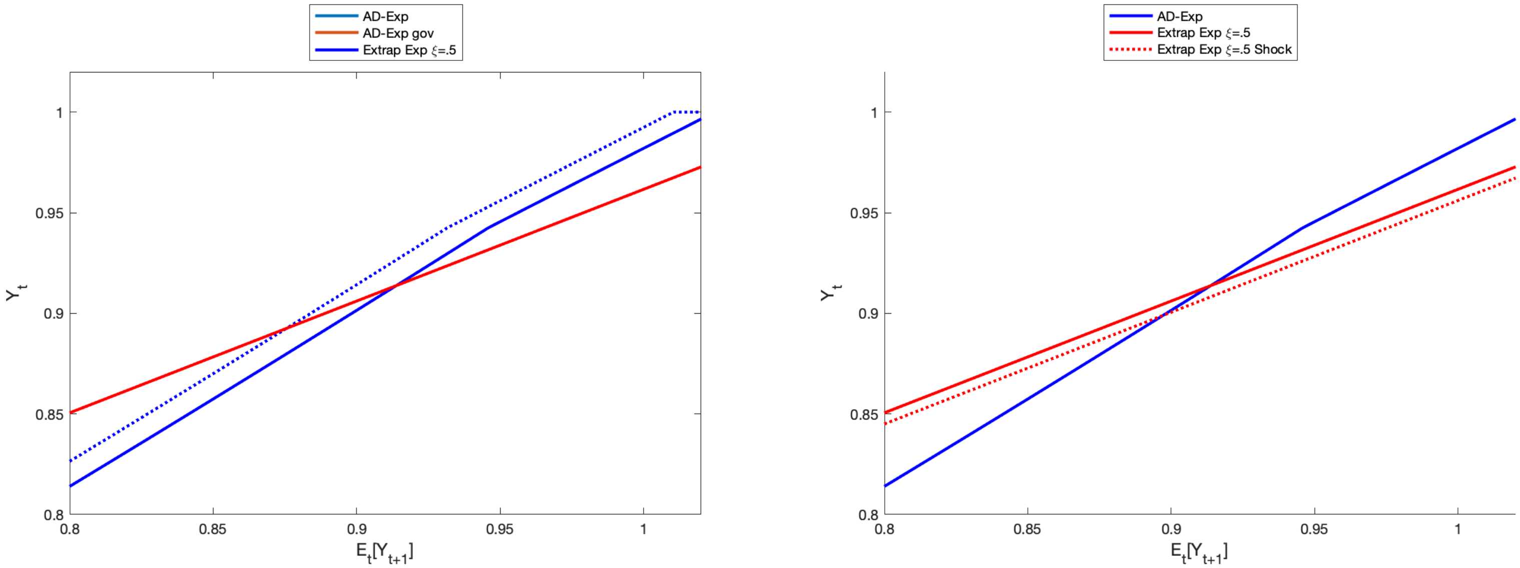
Figure A.4: Aggregate demand extrapolative expectations equilibrium, $\xi = .1$. No Neo-Fisherian properties.

Equilibrium in the model occurs at the intersection of the AD-Exp and Bel curves. At this point, markets clear, and future beliefs are consistent with the current level of output which generate these beliefs.

Whether the model has AK/NF properties depends on the relative slopes of the AD-Exp and Bel curves. In figure A.4a, the slope of the AD-Exp is less steep than than the slope of the Bel curve, and there are no Neo-Fisherian properties. Figure A.4a shows the effect of an increase in government spending: the AD-Exp curve shifts up, leading to an increase in both expectations and output. Figure A.4b shows the effect of a positive shock to expectations ϵ_t^e: the Bel curve shifts to the right, increasing both output and expectations.

Geometrically, it is clear that if slope of the AD-exp is less steep than the slope of the Bel curve, there are both no Neo-Fisherian properties, *and* a positive shock to beliefs leads to an increase in expectations in equilibrium. It is equally clear that that if the slope of the AD-Exp curve is steeper than the Bel curve, there will be both Neo-Fisherian properties and a positive shock to beliefs will lead to a decline in expectations.

In figure A.5a, we set $\xi = .25$, which lowers the slope of the Bel formation curve below that of the AD-Exp curve, and heralds the return of Neo-Fisherian properties. An increase in government spending in figure A.5a now *lowers* output and expectations of future output. Figure A.5b shows the effect of a positive shock to expectations, which leads to *lower* expectations and output.



(a) (b)

Figure A.5: Aggregate demand extrapolative expectations equilibrium, $\xi = .25$. Neo-Fisherian properties.

C List of Symbols

- Quantities
  - B_t^m, B_t^o: face value of real bonds for middle aged and old purchased at time $t-1$ at a price $1/\widehat{R_{t-1}}$
  - B_{t+1}^G: face value of real government bonds purchased at time t at price $1/\widehat{R_t}$
  - C_t^y, C_t^m, C_t^o: consumption of young, middle aged, old agents
  - $\widehat{C_{t+1}^o}$: subjective expectations of consumption in old age
  - G_t: government spending
  - L_t: labor demand/supply
  - π_t : Inflation
  - Π_t: profits of monopolistically competitive firms, distributed to middled aged Π_t^m and Π_t^o in proportion to γ
  - U_t: subjective utility of agents
  - $Spend_t(Y_t)$: aggregate expenditure function, which is a function of income
  - T_t: total taxes

- τ^y, τ^m, τ^o: fraction of taxes paid by the different generations.
- Y_t: income and output
- Y^*: full employment, equal to 1. Also referred to as Y^H.
- $\widehat{Y_{t+1}}$: subjective expectations of output
- $y_t^f(i)$: differentiated final goods in the CES aggregate, indexed by i.
- Y^L, Y^H: output in the high and low steady states.
- Y^0: output in the "zero" steady state
- Y_t^U: Unintended steady state in the Mertens & Ravn type model.
- $\overline{Y^U}$: in modified Mertens & Ravn model, agents expectation of output in unintended steady state.

• Prices

- I_t: gross nominal interest rate
- $p_t(i)$: price of differentiated final good, indexed by I
- P_t: nominal price index of final good aggregate. $P_t = \left(\int_0^1 p_t\,(i)^{1-\Lambda_t}\,di\right)^{\frac{1}{1-\Lambda_t}}$.
- $\widehat{\pi_{t+1}}$: subjective expectation of inflation
- π^*: central bank's inflation target
- π^H: inflation in the high steady state, equal to π^*
- π^L: inflation in the low steady state, equal to $1/R^*$.
- $\widehat{R_t}$: expected real interest rate, equal to $I_t/\widehat{\pi_{t+1}}$
- w_t: real wage

• Parameters

- $\alpha(\widehat{R_t})$: partial equilibrium Keynesian multiplier, equal to $\frac{1}{1-\overline{MPC}(\widehat{R_t})}$. Exact formula $\alpha(\widehat{R_t}) = \frac{1}{\left[1-mpc^y\frac{D}{\widehat{R_t}}-mpc^m\gamma-mpc^o(1-\gamma)\right]}$.
- D_t: debt limit for young agents.
- γ: fraction of labor supplied by middle age
- $I^* = (1+i^*)$: full employment nominal interest rate given inflation target, equal to $R^*\pi^*$.
- κ: slope of Phillips curve
- λ: adaptive learning updating parameter
- Λ: elasticity of substitution of CES aggregate
- $\overline{MPC}(\widehat{R_t}) = mpc^y\frac{D}{\widehat{R_t}} + mpc^m\gamma + mpc^o(1-\gamma)$. $\overline{MPC}(\widehat{R_t})$ is a weighted average of the individual generations' marginal propensities to consume. The weights are relative shares of income of the different generations.

- mpc^y, mpc^m, mpc^o: The marginal propensities to consume of the young, middle aged, and old are 1, $\frac{1}{1+\beta}$, and 1, respectively.
- μ: markup of monopolistically competitive firms, $\mu = (\Lambda/(\Lambda - 1)$.
- $\nu = \frac{\beta D}{1+\beta}$. This is the aggregate marginal propensity to consume out of income last period.
- ϕ_π: Taylor rule coefficient.
- $\Theta(\tau_t) = 1 - \tau_t^y - \frac{1}{1+\beta}\tau_t^m$. Parameter in IS-Exp curve with government spending, determines the size of government multiplier.
- $\Upsilon(\tau_t) = \tau_t^y + \frac{1}{1+\beta}\tau_t^m$. Parameter in IS-Exp curve with government spending, affects how government borrowing influences the multiplier.
- $\chi(\widehat{R_t}) = mpc^m(1-\gamma)/\widehat{R_t}$: the marginal propensity to consume out of expected future income.
- ξ: extrapolative expectations parameter.
- z: In Mertens & Ravn type model, the probability output will remain low.

* Shocks
  - ϵ_t^e: shock to expectations
  - ϵ_t^i: monetary policy shock to nominal interest rates